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ELEMENTS OF ALGEBRA

WITH EXERCISES

BY

GEORGE EGBERT FISHER, M.A., PH.D.

AND

ISAAC J. SCHWATT, PH.D.

ASSISTANT PROFESSORS OF MATHEMATICS IN THE
UNIVERSITY OF PENNSYLVANIA

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PREFACE.

THE unusual character of the recognition which the TEXT-BOOK OF ALGEBRA, Part I., has received encourages the authors to believe that a book on the same lines, but in briefer form, will have a still wider field of usefulness. This book retains the distinctive features of the larger volume; but it is in many respects, for younger students, an improvement on the latter.

The needs of beginners have been constantly kept in mind. The aim has been to make the transition from ordinary Arithmetic to Algebra natural and easy. No efforts have been spared to present the subject in a simple and clear manner. Yet nothing has been slighted or evaded, and all difficulties have been honestly faced and explained. New terms and ideas have been introduced only when the development of the subject made them necessary. Special attention has been paid to making clear the reason for every step taken. Each principle is first illustrated by particular examples, thus preparing the mind of the student to grasp the meaning of a formal statement of the principle and its proof. Directions for performing the different operations are, as a rule, given after these operations have been illustrated by particular examples.

The importance of mental discipline to every student of mathematics has also been fully recognized. On this account great care has been taken to develop the subject in a logical manner. Rigorous, but, as a rule, simple, proofs of all principles have been given.

If mathematics is to develop the reasoning power of the student and to teach him to think logically, it is better to omit a proof altogether than to give as a proof logically incorrect statements, thus training the mind of the student in illogical thinking.

Concrete illustrations, such as receiving and paying out money, going north and going south, have their proper places, but cannot be said to constitute proofs. If Algebra, like Arithmetic, treats of number, then the laws governing the operations with numbers should be derived from the properties of and the relations between them.

The subject-matter in the book has been printed in two sizes of type. The matter in smaller type consists of the formal proofs of principles and of the more difficult portions of each topic treated. The matter given in the larger type is logically complete (except for the proofs of principles), and can be taken up as a first course in the subject.

To economize space the exercises have been put in smaller type, and not the explanations and solutions of illustrative examples in the text. It is regarded as more important that the student should have these, which he is to study, most clearly represented rather than the examples which he is to copy and then work from his paper.

The attention of teachers is especially invited to the following features of the book:

The introductory chapter and the development in Chapter II. of the fundamental operations with algebraic numbers.

The use of type-forms in multiplication and division (Chapter VI.) and in factoring (Chapter VIII.).

The application of factoring to the solution of equations (Chapters VIII. and XXI.). By the early introduction of this method it has been possible to give problems which lead to quadratic equations before the formal treatment of that topic.

The solutions of equations based upon equivalent equations and equivalent systems of equations (Chapter IV., etc.). This method is of extreme importance, even to the beginner. The ordinary way of treating equations is illogical, leads to serious errors, and is therefore also pedagogically wrong.

Thus, no one will dispute that if both sides of an equation be multiplied by the same algebraical number an equation is obtained; but whether it is legitimate to assume that the solutions of this equation are the solutions of the given equation is quite another matter.

The treatment of irrational equations (Chapter XXIII.).

The special suggestions given in the first chapter on problems (Chapter V.), and applied subsequently to assist the student in acquiring facility in translating the verbal language of the problem into the symbolic language of the equation.

The discussion of general problems (Chapter XI.) and the interpretation of positive, negative, zero, indeterminate, and infinite solutions of problems (Chapter XII.).

The outline of irrational numbers (Chapter XVIII.).

The brief introduction to imaginary and complex numbers (Chapter XX.).

The exercises are voluminous. The aim has been not only to give examples for sufficient drill in the applications of the principles, but to include also many which tend to develop the thinking power of the student, rather than to develop him in a treadmill way.

Errors in the text and in the exercises may have been overlooked. Any suggestions from teachers and students with respect either to errors in the text and exercises or to the mode of presenting the subject will be highly appreciated.

The authors take pleasure in acknowledging their indebtedness to their colleagues in secondary schools and colleges for many helpful suggestions and criticisms which have been of much assistance to them in preparing this book.

The book is published in two forms:

(1) School Algebra. (2) Elements of Algebra.

The Elements contains the matter in the School Algebra and additional brief chapters on the more advanced subjects required for admission to universities and scientific schools.

G. E. F.
I. J. S.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, April, 1899.

PREFACE TO SECOND EDITION.

IN this edition a number of typographical errors have been corrected. We cordially thank those who have called any of them to our attention.

G. E. F.
I. J. S.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, August, 1899.

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CHAPTER I.

INTRODUCTION.

Algebra, like Arithmetic, treats of number. But the meaning of number, and the mode of representing it, are extended in passing from ordinary Arithmetic to Algebra.

§ 1. GENERAL NUMBER.

1. In ordinary Arithmetic all numbers have particular values and are represented by definite symbols, the Arabic numerals, 1, 2, 3, etc. The symbol 7, for instance, stands for a group of *seven* units.

In Algebra, however, such symbols as a, b, x, y, are used to represent numbers which may have *any values whatever,* or numbers whose values are, as yet, *unknown.*

Just as we speak of 10 miles, of 95 dollars, etc., in Arithmetic; so in Algebra we speak of a miles, meaning *any number* of miles or *an unknown number* of miles; of x dollars, meaning *any number* or *an unknown number* of dollars, etc.

For the sake of brevity, we shall say *the number* a, or simply a, meaning thereby *the number denoted by the symbol* a.

2. The symbols of Arithmetic, 1, 2, 3, etc., are retained in Algebra with their exact arithmetical meanings. The numbers represented by letters are, for the sake of distinction, called **Literal** or **General Numbers.** Other symbols than letters might be used to represent general numbers, but letters are more convenient to write and to pronounce.

3. The operations of Addition, Subtraction, Multiplication, and Division are denoted by the same symbols in Algebra as in Arithmetic.

4. The **Symbol of Addition**, $+$, read **plus**, is placed between two numbers to indicate that the number on its right is to be added to the number on its left.

E.g., just as $5+3$, read *five plus three*, means that 3 is to be added to 5; so $a+b$, read *a plus b*, means that b is to be added to a.

5. The **Symbol of Subtraction**, $-$, read **minus**, is placed between two numbers to indicate that the number on its right is to be subtracted from the number on its left.

E.g., just as $5-3$, read *five minus three*, means that 3 is to be subtracted from 5; so $a-b$, read *a minus b*, means that b is to be subtracted from a.

In a chain of additions and subtractions the operations are to be performed successively from left to right.

E.g., $7+4-3+2=11-3+2=8+2=10.$

6. The **Symbol of Multiplication**, $\times$, read **multiplied by**, or **times**, means that the number on its left is to be multiplied by the number on its right.

E.g., just as 5×3, read *five multiplied by three*, or *three times five*, means that 5 is to be multiplied by 3; so $a\times b$, read *a multiplied by b*, or *b times a*, means that a is to be multiplied by b.

A dot $(\cdot)$ is frequently used, instead of the symbol $\times$, to denote multiplication; as $a\cdot b$ for $a\times b$.

The symbol of multiplication between two literal numbers, or one literal number and an Arabic numeral, is frequently omitted.

E.g., the product $x\times y\times z$, or $x\cdot y\cdot z$, is usually written, xyz. The product $a\times 6$, or $a\cdot 6$, is written, $a6$.

It will be proved later that $a\times b=b\times a$, or $ab=ba$. On this account the product $a6$ is usually written $6a$, the Arabic numeral being placed first.

But the symbol of multiplication between two numerals cannot be omitted without changing the meaning.

E.g., if in the indicated multiplication, 3×6, or $3\cdot 6$, the symbol, $\times$, or $\cdot$, were omitted, we should have 36, not 18.

7. The **Symbol of Division**, $\div$, read **divided by**, is placed between two numbers to indicate that the number on its left is to be divided by the number on its right.

E.g., just as $10 \div 5$, read *ten divided by five*, means that 10 is to be divided by 5; so $a \div b$, read *a divided by b*, means that a is to be divided by b.

In a chain of multiplications and divisions the operations are to be performed successively from left to right.

E.g., $12 \times 2 \div 3 \times 4 = 24 \div 3 \times 4 = 8 \times 4 = 32$.

8. The use of letters to represent general numbers may be illustrated by a few simple examples.

Ex. 1. If a boy has 3 books and is given 2 more, he has $3 + 2$ books. If he has a books and is given 5 more, he has $a + 5$ books. If he has m books and is given n more, he has $m + n$ books.

Ex. 2. If a boy has 5 oranges and gives away 2, he has left $5 - 2$ oranges. If he has p oranges and gives away 7, he has left $p - 7$ oranges. If he has u oranges and gives away v, he has left $u - v$ oranges.

Ex. 3. If a man buys 5 city lots at 120 dollars each, he pays 120×5 dollars for the lots. If he buys a lots at 150 dollars each, he pays $150\,a$ dollars for the lots. If he buys u lots at v dollars each, he pays vu dollars for the lots.

Ex. 4. If a train runs 60 miles in 2 hours, it runs $60 \div 2$ miles in 1 hour. If it runs a miles in 5 hours, it runs $a \div 5$ miles in 1 hour. If it runs p miles in q hours, it runs $p \div q$ miles in 1 hour.

Ex. 5. If a pupil buys 2 note books at 10 cents each and 3 note books at 12 cents each, he pays $10 \times 2 + 12 \times 3$ cents for all. If he buys a note books at m cents each and b note books at n cents each, he pays $ma + nb$ cents for all.

Ex. 6. If, in a number of *two* digits, the digit in the *units'* place is 3 and the digit in the *tens'* place is 5, the number is $10 \times 5 + 3$. If the digit in the units' place is a and the digit in the tens' place is b, the number is $10\,b + a$.

Ex. 7. Just as $2 = 1 + 1$, and $3 = 1 + 1 + 1$,

so $2a = a + a$, and $3a = a + a + a$.

Therefore, just as $3 + 2 = 5$, so $3a + 2a = 5a$.

In like manner, $\frac{1}{2}x + \frac{2}{3}x = \frac{7}{6}x$.

9. Observe that in the preceding examples the reasoning is the same whether the numbers are represented by letters or by Arabic numerals. The results of these operations are *numbers* in all cases, whether letters or numerals, or both, are involved.

Thus, the result of adding b to a, $a + b$, is a number, just as $5 + 3$, or 8, is a number. Likewise, $a + b - c$, $ab - cd$, $3a - 5b$, $a \div b + a \div d$, etc., are numbers, *expressed by means of the signs and symbols of Algebra.*

EXERCISES I.

1. What number exceeds 7 by 3? What number exceeds 5 by a? What number exceeds x by 4? What number exceeds m by n?

2. The width of a room is a feet, and the length is b feet more than the width. What is the length of the room?

3. A man is now n years old. How old will he be in 20 years? How old in m years?

4. What number is less than 15 by 8? Less than a by 9? Less than 11 by b? Less than m by n?

5. A number N is divided into two unequal parts, the greater of which is 6; what is the less? If the less is a, what is the greater?

6. What number added to 16 gives 25? What number added to m gives n?

7. What number subtracted from 8 gives 5? What number subtracted from p gives q?

8. A man is n years old; how old was he 5 years ago? How old was he m years ago? How long must he live to be 90 years old? How long to be p years old?

9. What are the two *even* numbers nearest to 6, one greater and the other less than 6?

10. If m is an *even* number, what are the two nearest *even* numbers, one greater and the other less than m? The two nearest *odd* numbers, one greater and the other less than m?

11. If 1 pound of tea cost 75 cents, how much do 3 pounds cost? If 1 pound cost 75 cents, how much do n pounds cost? If 1 pound cost a cents, how much do b pounds cost?

12. If 3 men can do a piece of work in 8 hours, in how many hours can 1 man do the work? If a men can do a piece of work in 9 hours, in how many hours can 1 man do the work? If b men can do a piece of work in h hours, in how many hours can 1 man do the work?

13. 10×2, 10×3, etc., are *particular* multiples of 10; express *any* multiple of 10.

14. Write a number containing 5 units, 6 tens, 3 hundreds.

15. Write a number containing a units, b tens, c hundreds.

16. The speed of sound is 1100 feet per second. What is the distance of a cloud, if the thunder is heard 3 seconds after the flash of lightning? What is the distance of a cloud, if the thunder is heard b seconds after the flash?

17. If A rides a wheel 4 hours at the rate of 10 miles an hour, and B rides 3 hours at the rate of 14 miles an hour, how many miles do they both ride? How many more miles does B ride than A?

18. If A rides a wheel h hours at the rate of r miles an hour, and B rides k hours at the rate of s miles an hour, how many miles do they both ride? How many more miles does B ride than A?

19. By what number must 20 be multiplied to give 40? By what number must 20 be multiplied to give a? By what number must a be multiplied to give 20? By what number must n be multiplied to give b?

20. How many revolutions does a wheel 21 feet in circumference make in passing a distance of 35 yards? How many revolutions does a wheel c feet in circumference make in passing a distance of d yards?

21. A house costs a dollars, and rents for b dollars a month. What per cent does the investment pay?

22. If 22 yards of cloth cost $33, and 7 yards are sold for $14, what is the gain on each yard sold?

23. If d yards of cloth cost c dollars, and b yards are sold for a dollars, what is the gain on each yard sold?

10. Parentheses, (), and **Brackets,** [ ], are used to indicate that whatever is placed within them is to be treated as a whole.

E.g., $10 - (2 + 5)$ means that the result of adding 5 to 2, or 7, is to be subtracted from 10; that is,

$$10 - (2 + 5) = 10 - 7 = 3.$$

But $10 - 2 + 5$ means that 2 is to be subtracted from 10 and 5 is then to be added to that result; that is,

$$10 - 2 + 5 = 8 + 5 = 13.$$

In like manner, $[27-(3+2)\times 5]\div 2$ means that the result of multiplying the sum $3+2$ by 5 is first to be subtracted from 27, and the remainder is then to be divided by 2; that is,

$$[27-(3+2)\times 5]\div 2=[27-25]\div 2=2\div 2=1.$$

Likewise, the result of multiplying $a+b$ by c is $(a+b)c$, etc.

EXERCISES II.

Find the values of the following indicated operations:

1. $18+(7-3)$.
2. $12-(8-4)$.
3. $(25-11)-(18-7)$.
4. $(5+7)2$.
5. $12+(4-3)2$.
6. $(7+8)\div 5$.
7. $(12-6)\div 2$.
8. $25-(15-7)\div 2$.
9. $17-[(3+5)-(2+4)]$.
10. $[7+(11-2)-(8-6)]\times 2$.

11. What is the result of subtracting from x a number 5 greater than b?

12. One-third of a man's property is $a+100$ dollars. What is his entire property?

13. The length of a rectangular field is a rods, and its width is b rods less. What is the area of the field?

14. The older of two brothers is 20 years old; if he were 5 years younger, he would be three times as old as his younger brother. How old is his younger brother?

15. The older of two brothers is n years old; if he were a years younger, he would be b times as old as his younger brother. How old is the younger brother?

11. An **Algebraic Expression** is a number expressed by means of the signs and symbols of Algebra; as $ab-cd$, etc.

12. The **Symbol of Equality,** $=$, read *is equal to, has the value*, etc., is placed between two numbers or expressions to indicate that they have the same or equal values; as $3+2=5$.

An **Equation** is a statement that two numbers or expressions are equal; as $7\times 9=63$, $4\times 7+3=31$.

The *first*, or *left-hand member*, or *side*, of an equation is the expression on the *left* of the symbol $=$; the *second*, or *right-hand member*, or *side*, is the expression on the *right* of the symbol $=$.

13. The **Symbol of Inequality,** $>$, read *is greater than*, is used to indicate that the number or expression on its left is greater than that on its right; as $7>5$.

The **Symbol of Inequality,** $<$, read *is less than,* is used to indicate that the number or expression on its left is less than that on its right; as $3 < 4 + 2$.

Axioms.

14. An **Axiom** is a truth so simple that it cannot be made to depend upon a truth still simpler.

Algebra makes frequent use of the following mathematical axioms:

(i.) *Every number is equal to itself.* *E.g.*, $7 = 7$, $a = a$.

(ii.) *The whole is equal to the sum of all its parts.*

E.g., $7 = 3 + 4$, $5 = 1 + 1 + 1 + 1 + 1$.

(iii.) *If two numbers be equal, either can replace the other in any algebraic expression in which it occurs.*

E.g., If $a + b = c$, and $b = d$, then $a + d = c$, replacing b by d.

(iv.) *Two numbers which are each equal to a third number are equal to each other.*

E.g., If $a = b$, and $c = b$, then $a = c$.

(v.) *The whole is greater than any of its parts; and, conversely, any part is less than the whole.*

E.g., $3 + 2 > 2$ and $2 < 3 + 2$.

Fundamental Principles.

15. The following principles can be inferred directly from the axioms:

(i.) *If the same number, or equal numbers, be added to equal numbers, the sums will be equal.*

(ii.) *If the same number, or equal numbers, be subtracted from equal numbers, the remainders will be equal.*

(iii.) *If equal numbers be multiplied by the same number, or by equal numbers, the products will be equal.*

(iv.) *If equal numbers be divided by the same number (except 0), or by equal numbers, the quotients will be equal.*

16. *Literal numbers*, as has been stated, are used to represent numbers which may have *any values whatever*, or numbers whose values are, as yet, *unknown*. But it is frequently necessary to assign particular values to such numbers.

Substitution is the process of replacing a literal number in an algebraic expression by a particular value. See axiom (iii.).

Ex. 1. If in $a + b$, $a = 3$ (read a *has the value* 3) and $b = 5$, then

$$a + b = 3 + 5 = 8, \text{ or } a + b = 8.$$

Notice that the last step involved an application of axiom (iv.). For we have $a + b = 3 + 5$, and $3 + 5 = 8$; therefore, by axiom (iv.), $a + b = 8$.

Ex. 2. If in $a - (b + c)$, $a = 11$, $b = 2$, and $c = 3$, we have

$$a - (b + c) = 11 - (2 + 3) = 11 - 5 = 6.$$

Ex. 3. If, in $a + b - 2a + 3b - c$, we let $a = 6$, $b = 11\frac{1}{2}$, $c = \frac{5}{8}$, we have

$$\begin{aligned} a + b - 2a + 3b - c &= 6 + 11\tfrac{1}{2} - 2 \times 6 + 3 \times 11\tfrac{1}{2} - \tfrac{5}{8} \\ &= 6 + \tfrac{23}{2} - 12 + \tfrac{69}{2} - \tfrac{5}{8} = 38\tfrac{1}{8}. \end{aligned}$$

Observe that in the work of the last example, the expression $a + b - 2a + 3b - c$ is to be understood on the left of the symbol, =, in the second line.

Ex. 4. If, in the last example, $a = 3$, $b = 1$, and $c = 1$, we have $a + b - 2a + 3b - c = 3 + 1 - 6 + 3 - 1 = 4 - 6 + 3 - 1$.

We cannot further reduce $4-6+3-1$, since we are unable, *as yet*, to subtract 6 from 4.

EXERCISES III.

What are the values of the following expressions when $a = 6$, $b = 4$, $c = 2$:

1. $a + b$. **2.** $a - b$. **3.** ab. **4.** $a \div b$.

5. $a - b + c$. **6.** $a - b - c$. **7.** abc. **8.** $a \div b \times c$.

9. $a + (b - c)$. **10.** $a - (b + c)$. **11.** $(a - b)c$. **12.** $c \div (a - b)$.

13. $[a + (b - c)]a$. **14.** $[a - (b - c)] \div b$. **15.** $(a - b)(c - 1)$.

16. $(12 - a) \div (7 - b)$. **17.** $[15 - (7 - a)] \times [(25 - b) - (16 - c)]$.

17. Some of the advantages of using literal numbers are shown by the following example:

Ex. The two equations

$$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7} \text{ and } \frac{5}{11}+\frac{4}{11}=\frac{5+4}{11}=\frac{9}{11}$$

are particular examples of the following arithmetical principle:

The sum of two fractions which have a common denominator is a fraction whose denominator is that common denominator, and whose numerator is the sum of the two given numerators ; or,

$$\frac{\textit{1st num.}}{\textit{com. den.}}+\frac{\textit{2d num.}}{\textit{com. den.}}=\frac{\textit{1st num.}+\textit{2d num.}}{\textit{com. den.}}.$$

This principle can be stated still more concisely if the terms of the fractions, which may be *any numbers whatever*, are represented by three symbols, say a, b, c. We then have

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

This equation states by means of signs and symbols all that is contained in the verbal statement of the principle. It is thus a *symbolic statement of a general principle*, and includes all particular cases that result from assigning particular values to a, b, c.

18. Notice the following advantages thus secured by introducing general numbers:

(i.) *General laws and relations can be expressed with great brevity, and yet include all that the most general verbal statements can express.*

(ii.) *Such symbolic statements mass under the eye the various operations involved, and thus enable the eye to assist the understanding and memory.*

EXERCISES IV.

Express *in algebraic language* (*i.e.*, by means of the signs and symbols of Algebra) the following principles of Arithmetic:

1. If a, b, and c are any three numbers, their sum diminished by any one of them is equal to the sum of the other two.

If z is the result of subtracting y from x, express in algebraic language the following principles of subtraction:

2. The minuend is equal to the subtrahend plus the remainder.

3. The subtrahend is equal to the minuend diminished by the remainder.

If z is the result of multiplying x by y, express in algebraic language the following principles of multiplication:

4. The multiplicand is equal to the product divided by the multiplier.

5. The multiplier is equal to the product divided by the multiplicand.

If a is exactly divisible by b, and q is the quotient, express in algebraic language the following principles of division:

6. The dividend is equal to the divisor multiplied by the quotient.

7. The divisor is equal to the dividend divided by the quotient.

If $\frac{a}{b}$ is any fraction, and m is any integer, express in algebraic language the following principles of fractions:

8. If the numerator of a fraction is multiplied by any integer, the value of the fraction is multiplied by that integer.

9. If the denominator of a fraction is multiplied by any integer, the value of the fraction is divided by that integer.

Problems solved by Equations.

19. Another advantage of using literal numbers is shown by the following problem:

Pr. The older of two brothers has twice as many marbles as the younger, and together they have 33 marbles. How many has the younger?

The number of marbles the younger brother has is, as yet, an *unknown number.* Let us represent this unknown number by some letter, say x. Then, since the older brother has twice as many, he has $x \times 2$, or $2x$, marbles. The problem states, in *verbal* language: *the number of marbles the younger has plus the number the older has is equal to 33;*

in *algebraic* language: $x + 2x = 33$, or $3x = 33$.

Dividing by 3 [Art. 15 (iv.)], $x = 11$, the number of marbles the younger has. The older has $2x, = 22$.

Check: $x + 2x = 11 + 22 = 33.$

EXERCISES V.

1. What number added to three times itself gives 28?

2. Divide 69 into two parts so that the greater shall be twice the less.

3. If twice a number be added to three times the number, the sum will be 63. What is the number?

4. If three times a number be subtracted from five times the number, the remainder will be 6. What is the number?

5. Divide 150 into two parts so that the less is one-fifth of the greater.

6. A and B together have $180, and A has five times as much as B. How many dollars has each?

7. In a school are 120 pupils; in the second grade are twice as many as in the first, and in the third three times as many as in the first. How many pupils are in each grade?

8. A, B, and C together invest $8000; A invests twice as much as B, and B five times as much as C. How many dollars does each invest?

9. Divide 30 into three parts, so that the second shall be one-half of the first, and the third one-third of the second.

10. Divide 52 into three parts, so that the second shall be one-half of the first, and the third one-fourth of the second.

11. Divide 85 into three parts, so that the first shall be four times the second, and one-third of the third.

12. If one-fourth of a number be subtracted from one-third of the number, the remainder will be 6. What is the number?

13. Three boys, A, B, and C, have together 27 pencils. B has twice as many as A, and C twice as many as A and B together. How many pencils has each?

14. Three boys, A, B, and C, have together 32 pens; B has one-third as many as A, and C three times as many as A and B together. How many has each?

15. Three boys, A, B, and C, have together 16 note books; A has five times as many as B, and the number that A has more than B is twice the number that C has. How many note books has each?

20. General numbers are most frequently represented by the *italicized letters* of the English alphabet. But letters of other alphabets are sometimes employed, and there is often an advantage in using the same letter with some distinguishing marks to represent different numbers in the same discussion.

We add a list of the more common symbols.

Greek letters: α, β, γ, δ, etc., read *alpha, beta, gamma, delta,* etc.;

with *prime marks:* a', a'', a''', $a^{(n)}$, read *a prime, a two prime, a three prime, a n prime;*

with *subscripts:* a_1, a_2, a_3, etc., read *a sub-one, a sub-two, a sub-three,* etc., or simply *a one, a two, a three,* etc.

§ 2. POSITIVE AND NEGATIVE NUMBERS, OR ALGEBRAIC NUMBERS.

1. A still greater extension of the idea of number in passing from Arithmetic to Algebra is arrived at by the following considerations:

In ordinary Arithmetic we subtract a number from a greater or an equal number. We are familiar with such operations as

$$7-5=2, \quad 6-5=1, \quad 5-5=0. \qquad \text{(i.)}$$

But such operations as

$$4-5, \quad 3-5, \text{ etc.,} \qquad \text{(ii.)}$$

have not occurred in ordinary Arithmetic and cannot be carried out in terms of arithmetical numbers. For, from an arithmetical point of view, we cannot subtract from a number more units than are contained in that number. In general, the indicated operation $a-b$ can, as yet, be performed only when a is greater than b. But if a and b are to have any values whatever, the case in which a is less than b, that is, in which *the minuend is less than the subtrahend,* must be included in the operation of subtraction.

2. Now observe that, *as the minuend in equations* (i.) *decreases by* 1, 2, *or more units* (*the subtrahend remaining the same*) *the remainder decreases by an equal number of units.* When the minuend is equal to the subtrahend, the remainder is 0. If then, as in the indicated operations (ii.), the minuend becomes less than the subtrahend by 1, 2, or more units, the remainder must decrease by an equal number of units, and therefore become less than 0 by 1, 2, or more units.

The operation of subtracting a greater number from a less is therefore possible only when numbers less than zero are introduced.

We then have from (i.) and (ii.):

$$\left.\begin{array}{l}\text{Min.} - \text{Subt.} = \text{Rem.}\\ 7-5=2\\ 6-5=1\\ 5-5=0\\ 4-5=\textit{a number one unit less than } 0\\ 3-5=\textit{a number two units less than } 0\end{array}\right\}\quad\text{(iii.)}$$

3. *Numbers less than zero* are called **Negative Numbers.** *Numbers greater than zero* are, for the sake of distinction, called **Positive Numbers.**

Positive and negative numbers are called **Algebraic** or **Relative Numbers.**

4. The **Absolute Value** of a number is the number of units contained in it without regard to their *quality* (*i.e. whether positive or negative*).

A *positive* number may be indicated by placing a small sign, $^+$, to the left and a little above its absolute value; as, $^+5$, $^+10$, $^+16$; read *positive* 5, *positive* 10, *positive* 16.

A *negative* number may be indicated by placing a small sign, $^-$, to the left and a little above its absolute value; as, $^-5$, $^-10$, $^-16$; read *negative* 5, *negative* 10, *negative* 16.

We must, as yet, carefully distinguish these symbols of *quality*, $^+$ and $^-$, from the (larger) symbols of *operation*, $+$ and $-$.

5. Equations (iii.) can now be written as follows:

$$\left.\begin{array}{l}\text{Min.} - \text{Subt.} = \text{Rem.}\\ \textit{pos. } 7-\textit{pos. } 5=\textit{pos. } 2\\ \textit{pos. } 6-\textit{pos. } 5=\textit{pos. } 1\\ \textit{pos. } 5-\textit{pos. } 5= 0\\ \textit{pos. } 4-\textit{pos. } 5=\textit{neg. } 1\\ \textit{pos. } 3-\textit{pos. } 5=\textit{neg. } 2\end{array}\right\}\ \text{or}\ \left\{\begin{array}{l}\text{Min.} - \text{Subt.} = \text{Rem.}\\ ^+7-{^+5}={^+2}\\ ^+6-{^+5}={^+1}\\ ^+5-{^+5}=0\\ ^+4-{^+5}={^-1}\\ ^+3-{^+5}={^-2}\end{array}\right\}\quad\text{(iv.)}$$

A negative remainder does not mean that more units have been taken from the minuend than were contained in it; *such* *a remainder indicates that the subtrahend is greater than the minuend by as many units as are contained in the remainder.*

Thus, in ${}^{+}10 - {}^{+}15 = {}^{-}5$ and ${}^{+}87 - {}^{+}92 = {}^{-}5$, the remainder, ${}^{-}5$, indicates that the subtrahend is, in each case, 5 units greater than the minuend.

6. The results of the preceding articles, restated briefly, are:

A positive number is a number greater than zero, by as many units as are contained in its absolute value.

E.g., ${}^{+}2$ is *two* units greater than 0.

A negative number is a number less than zero by as many units as are contained in its absolute value.

E.g., ${}^{-}3$ is *three* units less than 0.

Zero is the result of subtracting a number from an equal number.

E.g., $0 = {}^{+}7 - {}^{+}7 = {}^{-}5 - {}^{-}5 = {}^{+}n - {}^{+}n = {}^{-}n - {}^{-}n$,

wherein n denotes any absolute number.

Since zero can be neither greater nor less than itself, it is neither a positive nor a negative number. It stands by itself, as the number from which positive and negative numbers are counted.

7. The **Sign of Continuation**, $\cdots$, read *and so on*, or *and so on to*, is used to indicate that a succession of numbers continues without end, as 1, 2, 3, $\cdots$, read, *one, two, three, and so on;* or that the succession continues as far as a certain number which is written after the sign $\cdots$, as 1, 2, 3, $\cdots$, 10, read *one, two, three, as far as*, or *to*, 10.

We may now write the series of algebraic numbers:

$$\cdots {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, {}^{+}4, \cdots$$

In this series the numbers increase from left to right, and decrease from right to left; or a number is greater than any number on its left and less than any number on its right.

The numbers of ordinary Arithmetic are the absolute values of the positive and negative numbers of Algebra.

Relations between Positive and Negative Numbers and Zero.

8. From the results of the preceding article, we obtain the following general relations:

(i.) *Of two positive numbers, that number is the greater which has the greater absolute value; and that number is the less which has the less absolute value.*

(ii.) *Of two negative numbers, that number is the greater which has the less absolute value; and that number is the less which has the greater absolute value.*

For example, $^-3 > {}^-5$, or $^-5 < {}^-3$, since $^-5$ is five units less than 0, and $^-3$ is only three units less than 0.

9. Although negative numbers arise through the extension of the operation of subtraction, it is necessary to treat them as numbers apart from this particular operation.

As in Arithmetic, so in Algebra, any integer is an aggregate of like units.

Just as $$4 = 1 + 1 + 1 + 1,$$
so $$^+4 = {}^+1 + {}^+1 + {}^+1 + {}^+1, \text{ and } {}^-4 = {}^-1 + {}^-1 + {}^-1 + {}^-1.$$
Just as $$\tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3},$$
so $$^+(\tfrac{2}{3}) = {}^+(\tfrac{1}{3}) + {}^+(\tfrac{1}{3}), \text{ and } {}^-(\tfrac{2}{3}) = {}^-(\tfrac{1}{3}) + {}^-(\tfrac{1}{3}).$$

Since letters are to represent numbers which may have any values whatever, they can represent either *positive* or *negative* numbers. Thus, in one case a may have the value $^+2$, in another case the value $^-7$; in the first case the absolute value of a is 2, in the second case the absolute value of a is 7.

EXERCISES VI.

1. What is the absolute value of $^+8$? Of $^-11$? Of $^+(2 + y)$?

For what values of x do the following expressions reduce to 0:

2. $x - {}^+3$? **3.** $x - {}^-18$? **4.** $x - {}^+a$? **5.** $x - {}^-(a + 6)$?

What values of a make the first members of the following equations identical with the second members:

6. $a - {}^+7 = {}^+2$? **7.** $a - {}^+7 = {}^-2$? **8.** $a - {}^+7 = {}^-5$?

What are the results of the following indicated operations:

9. $^+17 - {}^+2$? **10.** $^+2 - {}^+30$? **11.** $^+19 - {}^+25$? **12.** $^+(\tfrac{1}{2}) - {}^+(\tfrac{3}{4})$?

13. $^+(a + 2) - {}^+2$? **14.** $^+n - {}^+(n + 3)$? **15.** $^+9 - {}^+(n + 9)$?

How many units is each of the following numbers greater or less than 0:

16. ${}^{+}10$? **17.** ${}^{-}3$? **18.** ${}^{-}(\frac{3}{4})$? **19.** ${}^{+}x$? **20.** ${}^{-}y$?

Which number is greater,

21. ${}^{+}3$ or ${}^{-}5$? **22.** ${}^{-}12$ or ${}^{-}5$? **23.** 0 or ${}^{-}3$? **24.** ${}^{-}5$ or ${}^{+}4$?

25. ${}^{+}(5+a)$ or ${}^{+}a$? **26.** ${}^{-}(5+a)$ or ${}^{-}a$? **27.** ${}^{-}(6+a)$ or ${}^{+}(2+a)$?

28. ${}^{-}(a+1)$ or ${}^{-}(a-1)$? **29.** ${}^{-}(a+1)$ or ${}^{+}(a-1)$?

Positive and Negative Numbers are Opposite Numbers.

10. The student is familiar with the principle of subtraction in Arithmetic that *the remainder added to the subtrahend is equal to the minuend.* This principle, like all principles of arithmetical operations, is retained in Algebra. Consequently, continuing equations (iv.), Art. 5, we have:

$$\begin{array}{c}\text{Min.} - \text{Subt.} = \text{Rem.}\\ \left.\begin{array}{r}{}^{+}3 - {}^{+}5 = {}^{-}2\\ {}^{+}2 - {}^{+}5 = {}^{-}3\\ {}^{+}1 - {}^{+}5 = {}^{-}4\\ 0 - {}^{+}5 = {}^{-}5\end{array}\right\}\end{array} \text{(v.), and } \begin{array}{c}\text{Subt.} + \text{Rem.} = \text{Min.}\\ \left.\begin{array}{r}{}^{+}5 + {}^{-}2 = {}^{+}3\\ {}^{+}5 + {}^{-}3 = {}^{+}2\\ {}^{+}5 + {}^{-}4 = {}^{+}1\\ {}^{+}5 + {}^{-}5 = 0\end{array}\right\}\end{array} \text{(vi.)}$$

11. The last of equations (vi.), ${}^{+}5 + {}^{-}5 = 0$, furnishes an important relation between positive and negative numbers:

The sum of a positive and a negative number having the same absolute value is equal to zero; i.e., *two such numbers cancel each other when united by addition.*

E.g., ${}^{+}1 + {}^{-}1 = 0$, ${}^{+}3 + {}^{-}3 = 0$, ${}^{-}17\frac{1}{2} + {}^{+}17\frac{1}{2} = 0$.

In general, ${}^{+}n + {}^{-}n = 0$.

For this reason, positive and negative numbers in their relation to each other are called *opposite* numbers. When their absolute values are equal, they are called *equal* and *opposite* numbers.

12. Any *quantities* which in their relation to each other are *opposite,* may be represented in Algebra by *positive* and *negative* numbers; as *credits* and *debits, gain* and *loss.*

Ex. 1. 100 *dollars credit* and 100 *dollars debit* cancel each other. That is, 100 *dollars credit united with* 100 *dollars debit is equal to neither credit nor debit;* or, *100 dollars credit* + *100 dollars debit* = *neither credit nor debit.*

If *credits* be taken *positively* and *debits negatively*, then 100 *dollars credit* may be represented by $^+100$, and 100 *dollars debit* by $^-100$. Their united effect, as stated above, may then be represented algebraically thus:

$$^+100 + {}^-100 = 0.$$

The result, 0, means *neither credit nor debit.*

Similarly for *opposite temperatures.*

Ex. **2.** If a body is first heated 10° and then cooled down 8°, its final temperature is 2° above its original temperature; or, stated algebraically,

$$^+10 + {}^-8 = {}^+2.$$

The result, $^+2$, means a *rise* of 2° in temperature.

Similar reasoning applies to *opposite directions.*

Ex. **3.** If a man walks 10 miles due *north*, and turning, walks 14 miles due *south*, he is then 4 miles *south* of his starting point; or,

$$^+10 + {}^-14 = {}^-4.$$

The result, $^-4$, means that he is now 4 miles *south* of his starting point.

13. It is evidently immaterial which of two opposite quantities is taken positively and which negatively in any particular problem. Thus, we might call distances *south positive* and distances *north negative.* We have only to interpret results differently.

EXERCISES VII.

State algebraically in two ways each of the following relations (by Art. 13):

1. 100 dollars gain and 20 dollars loss is equivalent to 80 dollars net gain.

2. 250 dollars gain and 250 dollars loss is equivalent to neither gain nor loss.

3. A rise of 15° in temperature followed by a fall of 22° is equivalent to a fall of 7°.

4. If a man ascends from the foot of a ladder 20 steps, and then descends 7 steps, he is 13 steps up.

5. If a man ascends from the foot of a ladder 10 steps, and then descends 10 steps, he is at the foot of the ladder.

6. If a man walks 150 feet to the right, then 50 feet to the left, and then 75 feet to the right, he is finally 175 feet to the right of his original position.

7. Two men, A and B, run a race. The first minute A runs 5 feet more than B, the second minute A runs 8 feet less than B; in the two minutes A runs 3 feet less than B.

14. We have defined negative numbers as numbers less than zero; that is, as the result of enlarging our conception of the operation of subtraction. We afterward find, as we have seen, that they often have a meaning when applied to practical problems. Yet even if they had not, we should be justified in introducing them in order to make our principles general. Sometimes, indeed, negative results indicate an impossibility.

E.g., a men are at work on a building, and b men quit work. How many are still working? Evidently $a-b$. If $b>a$, the negative result has no meaning, and indicates that we have stated an impossibility.

A man has a dollars and pays out b dollars. How many dollars has he left? Evidently $a-b$.

When $b>a$, the negative result has no meaning, if the money be regarded as actually handled. But in dealing with book accounts, it is quite possible that the debits shall exceed the credits; a state which would, as we have seen, be indicated by a negative result (if credits be taken positively and debits negatively). So, too, when applied to opposition in direction, etc., negative results are as intelligible as positive results.

In fact, there is no more objection to the use of negative numbers than to the use of fractions, for each kind of number may indicate an impossible state. For instance, there are a men in a company, which is divided into b equal groups. How many men are there in each group? Evidently $a \div b$. If a be not exactly divisible by b, the result is as impossible as taking b men from a men, when $b>a$.

In Arithmetic we proceed to prove all the laws of fractions, without inquiring whether they can be applied in all cases. So, in Algebra, we shall proceed to operate with and upon negative numbers without inquiring whether or not they will always have a meaning in particular problems.

CHAPTER II.

THE FOUR FUNDAMENTAL OPERATIONS WITH ALGEBRAIC NUMBER.

§ 1. ADDITION OF ALGEBRAIC NUMBERS.

1. **Addition** *of one algebraic number to another is the process of uniting it with the other into one aggregate.*

As in Arithmetic, the one number is said to be **added** to the other, and the result of the addition is called the **Sum.**

Addition of Numbers with Like Signs.

2. Ex. **1.** Add $^+3$ to $^+4$.

The three positive units, $^+3$, when united by addition with the four positive units, $^+4$, give an aggregate of *four plus three*, or *seven*, positive units. That is,

$$^+4 + {^+3} = {^+(4 + 3)} = {^+7}.$$

In like manner

Ex. **2.** $^-4 + {^-3} = {^-(4 + 3)} = {^-7}$.

Ex. **3.** $^+2 + {^+(\frac{3}{4})} = {^+(2 + \frac{3}{4})} = {^+2\frac{3}{4}}$.

These examples illustrate the following principle:

To add one algebraic number to another, with like sign (i.e., *both numbers positive or both negative*), *add arithmetically the absolute value of the one number to the absolute value of the other, and prefix to the sum the common sign of quality.* Or, stated symbolically,

$$^+a + {^+b} = {^+(a + b)} \quad \text{(i.).} \qquad {^-a} + {^-b} = {^-(a + b)} \quad \text{(ii.).}$$

Addition of Numbers with Unlike Signs.

3. Ex. **1.** Add $^-2$ to $^+5$.

The two negative units, $^-2$, when united by addition with the five positive units, $^+5$, *cancel two of the five positive units*

(Chap. I., § 2, Art. 11). There remain then *five minus two,* or *three,* positive units. That is,

$$^{+}5 + {}^{-}2 = {}^{+}(5 - 2) = {}^{+}3.$$

Ex. **2.** Add $^{+}2$ to $^{-}5$.

The two positive units, $^{+}2$, when united by addition with the five negative units, $^{-}5$, *cancel two of the five negative units.* There remain then *five minus two,* or *three,* negative units.

That is, $$^{-}5 + {}^{+}2 = {}^{-}(5 - 2) = {}^{-}3.$$

Observe that in both examples *the sum is of the same quality as the number which has the greater absolute value, and that the absolute value of the sum is obtained by subtracting the less absolute value,* 2, *from the greater,* 5.

Ex. **3.** $^{+}2 + {}^{-}(\frac{3}{4}) = {}^{+}(2 - \frac{3}{4}) = {}^{+}1\frac{1}{4}$.

These examples illustrate the following principle:

To add one algebraic number to another, with unlike sign, subtract arithmetically the less absolute value from the greater, and prefix to the remainder the sign of quality of the number which has the greater absolute value. Or, stated symbolically,

$$^{+}\boldsymbol{a} + {}^{-}\boldsymbol{b} = {}^{+}(\boldsymbol{a} - \boldsymbol{b}),\text{ when } \boldsymbol{a} > \boldsymbol{b} \quad \text{(iii.)},$$

$$^{+}\boldsymbol{a} + {}^{-}\boldsymbol{b} = {}^{-}(\boldsymbol{b} - \boldsymbol{a}),\text{ when } \boldsymbol{a} < \boldsymbol{b} \quad \text{(iv.)}.$$

4. The proofs of principles (i.)-(iv.) are as follows:

In (i.), the positive units and parts of positive units represented by ^{+}b, when united by addition with the positive units and parts of positive units represented by ^{+}a, give an aggregate of positive units and parts of positive units represented by $^{+}(a + b)$. In like manner (ii.) can be proved.

In (iii.), the negative units and parts of negative units represented by ^{-}b, when united by addition with the positive units and parts of positive units represented by ^{+}a, cancel an equal number of positive units and parts of positive units. There remain positive units and parts of positive units represented by $^{+}(a - b)$. In like manner (iv.) can be proved.

Addition of Three or More Numbers.

5. To unite three or more algebraic numbers by addition, add the second to the first, to that sum add the third, again to that sum the fourth, and so on.

Ex. **1.** $^{+}2 + {}^{+}3 + {}^{+}7 = {}^{+}5 + {}^{+}7 = {}^{+}12.$

Ex. **2.** $^{+}11 + {}^{-}8 + {}^{+}2 = {}^{+}3 + {}^{+}2 = {}^{+}5.$

EXERCISES I.

Find the results of the following indicated additions:

1. $^+11 + {}^+5$. **2.** $^-9 + {}^-5$. **3.** $^-2\frac{2}{3} + {}^-3\frac{1}{3}$. **4.** $^-16 + {}^+7$.

5. $^+16 + {}^-7$. **6.** $^-3\frac{3}{8} + {}^+1\frac{1}{4}$. **7.** $^+5\frac{1}{2} + {}^-2\frac{3}{4}$. **8.** $0 + {}^-5$.

9. $^+7 + {}^-5 + {}^+8$. **10.** $^-8 + {}^+11 + {}^-3$. **11.** $^-81 + {}^+70 + {}^-180 + {}^+12$.

Add

12. $^+17$ to $^+5$. **13.** $^-6$ to $^-27$. **14.** $^+13$ to ^+a. **15.** $^-18$ to ^-b.

16. $^+10$ to $^-5$. **17.** $^-20$ to $^+6$. **18.** $^+20$ to $^-6$. **19.** $^-(\frac{1}{2})$ to $^+(\frac{5}{6})$.

20. $^+11$ to ^-a, when $a > 11$. **21.** $^+11$ to ^-a, when $a < 11$.

22. $^-17$ to ^+x, when $x > 17$. **23.** $^-17$ to ^+x, when $x < 17$.

24. $^-2$ to $^+8 + {}^-4$. **25.** $^+18$ to $^-2 + {}^+35$. **26.** $^-5$ to $^-11 + {}^+15$.

Find the results of the following indicated additions, first uniting the numbers within the parentheses:

27. $^+7 + ({}^+8 + {}^-3)$. **28.** $^+11 + ({}^-12 + {}^+2)$.

29. $({}^+2 + {}^-3) + ({}^-11 + {}^+12)$. **30.** $({}^+5 + {}^-8) + ({}^-12 + {}^+3)$.

What is the value of $a + b$,

31. When $a = {}^+5$, $b = {}^+3$? **32.** When $a = {}^-7\frac{7}{8}$, $b = {}^-3\frac{3}{4}$?

33. When $a = {}^+71$, $b = {}^-53$? **34.** When $a = {}^+25$, $b = {}^-34$?

35. When $a = {}^+2 + {}^-3$, $b = {}^-8 + {}^+7$?

36. When $a = {}^-5 + {}^+3$, $b = {}^+11 + {}^-4$?

What is the value of $a + b$, wherein $a = m + n$ and $b = p + q$,

37. When $m = {}^-1$, $n = {}^+2$, $p = {}^-3$, $q = {}^+4$?

38. When $m = {}^+8$, $n = {}^-3$, $p = {}^-5$, $q = {}^-9$?

The Associative and Commutative Laws for Addition.

6. In the preceding articles the process of addition has been carried out from left to right from number to number.

E.g., $^+7 + {}^-3 + {}^+5 + {}^-8 = {}^+4 + {}^+5 + {}^-8 = {}^+9 + {}^-8 = {}^+1$.

But the result is the same if two or more successive numbers be *associated* in performing the additions.

E.g., $^+7 + {}^-3 + ({}^+5 + {}^-8) = {}^+7 + {}^-3 + {}^-3 = {}^+4 + {}^-3 = {}^+1$.

$^+7 + ({}^-3 + {}^+5) + {}^-8 = {}^+7 + {}^+2 + {}^-8 = {}^+9 + {}^-8 = {}^+1$.

This example illustrates the following principle:

The Associative Law. — *The sum of three or more numbers is the same in whatever way successive numbers are grouped or associated in the process of adding.* Or, stated symbolically,

$$a + b + c = a + (b + c);$$

$$a+b+c+d=a+b+(c+d)=a+(b+c+d)=a+(b+c)+d.$$

7. In an indicated addition, the number on the right of the sign $+$ is to be added to the number on its left.

E.g., In $\quad {}^{+}5 + {}^{-}3, = {}^{+}2,$

${}^{-}3$ is added to ${}^{+}5$; while in

$${}^{-}3 + {}^{+}5, = {}^{+}2,$$

${}^{+}5$ is added to ${}^{-}3$. But the result is the same, whichever of the two numbers, ${}^{+}5$ and ${}^{-}3$, be added to the other. That is,

$${}^{+}5 + {}^{-}3 = {}^{-}3 + {}^{+}5.$$

This example illustrates the following principle:

The Commutative Law. — *The sum of two or more numbers is the same in whatever order they may be added.* Or, stated symbolically,

$$a + b = b + a$$

$$\begin{aligned} a + b + c + d &= b + a + d + c \\ &= d + c + b + a, \text{ etc.} \end{aligned}$$

8. The proof of the principles enunciated in Arts. 6 and 7 is as follows: The total number of units and parts of units, positive and negative, in the given numbers is the same in whatever way they may be grouped or arranged; a given number of positive units will cancel an equal number of negative units, and *vice versa;* and a given number of parts of positive units will cancel an equal number of like parts of negative units, and *vice versa.* Therefore the final result will be the same, whatever order or way of associating the units and parts of units may be used.

9. Since $a + b = b + a$, the two numbers a and b, when united by addition, are given the common name **Summand.**

10. The Associative and Commutative Laws may be applied simultaneously.

E.g., ${}^{-}2 + {}^{+}4 + {}^{-}3 + {}^{+}1 = ({}^{-}2 + {}^{-}3) + ({}^{+}4 + {}^{+}1) = {}^{-}5 + {}^{+}5 = 0.$

In general, $a + b + c = a + (c + b) = c + (a + b)$, etc.

11. In adding three or more numbers, some of which are positive and some negative, the Commutative and Associative Laws enable us to employ the following method:

Add all the numbers of one sign, then all the numbers of the opposite sign, and add the two resulting sums.

E.g.,

$$^{-}8 + {}^{+}3 + {}^{-}5 + {}^{+}7 + {}^{+}3 = {}^{-}8 + {}^{-}5 + {}^{+}3 + {}^{+}7 + {}^{+}3 = {}^{-}13 + {}^{+}13 = 0.$$

EXERCISES II.

Find, in three different ways, by applying the Commutative Law, the values of:

1. $^{+}18 + {}^{-}4 + {}^{+}2$. **2.** $^{+}12 + {}^{-}13 + {}^{+}1$. **3.** $^{-}20 + {}^{-}3 + {}^{+}17$.

Find, in the most convenient way, the values of:

4. $^{-}998 + {}^{+}500 + {}^{-}2$. **5.** $^{+}333\frac{1}{3} + {}^{-}125 + {}^{+}66\frac{2}{3}$.

Find, in the most convenient way, the value of $a + b + c + d$,

6. When $a = {}^{-}5$, $b = {}^{+}100$, $c = {}^{-}95$, $d = {}^{+}4$.

7. When $a = {}^{-}763$, $b = {}^{+}1000$, $c = {}^{-}237$, $d = {}^{-}3$.

Find the values of the following expressions by the method of Art. 11:

8. $^{+}3 + {}^{-}4 + {}^{-}6 + {}^{+}9 + {}^{+}2$. **9.** $^{-}5 + {}^{+}7 + {}^{+}19 + {}^{-}15 + {}^{-}22$.

10. $^{-}13 + {}^{+}5 + {}^{-}15 + {}^{+}8 + {}^{-}4$. **11.** $^{-}(\frac{2}{3}) + {}^{+}(\frac{5}{6}) + {}^{-}(\frac{7}{8}) + {}^{-}(\frac{5}{9}) + {}^{+}(\frac{1}{3})$.

12. In ordinary Arithmetic to add a number to any number increases the latter.

E.g., $7 + 4 = 11$, and $11 > 7$.

But such is not always the case in adding one algebraic number to another.

E.g., $^{+}7 + {}^{+}4 = {}^{+}11$, and $^{+}11 > {}^{+}7$;

but $^{+}7 + {}^{-}4 = {}^{+}3$, and $^{+}3 < {}^{+}7$.

Property of Zero in Addition.

13. We have $^{+}3 + {}^{+}2 + {}^{-}2 = {}^{+}3$.

But $^{+}3 + {}^{+}2 + {}^{-}2 = {}^{+}3 + ({}^{+}2 + {}^{-}2)$, by Assoc. Law,

$= {}^{+}3 + 0$, since $^{+}2 + {}^{-}2 = 0$.

Therefore, by Axiom (iv.), $^{+}3 + 0 = {}^{+}3$.

In general, $N + {}^{+}a + {}^{-}a = N$.

But $N + {}^{+}a + {}^{-}a = N + ({}^{+}a + {}^{-}a) = N + 0$.

Therefore, by Axiom (iv.), $\boldsymbol{N + 0 = N}$. (i.)

§ 2. SUBTRACTION OF ALGEBRAIC NUMBERS.

1. Subtraction *is the inverse of addition.* In addition two numbers are given, and it is required to find their sum. In subtraction the sum and one of the numbers are given, and it is required to find the other number.

As in ordinary Arithmetic, the given sum is called the **Minuend,** the given number the **Subtrahend,** and the required number the **Remainder.**

Ex. 1. Subtract ${}^{-}2$ from ${}^{+}9 + {}^{-}2$.

We have $({}^{+}9 + {}^{-}2) - {}^{-}2 = {}^{+}9$, by definition of subtraction.

That is, *if from the sum of two numbers either of the numbers be subtracted, the remainder is the other number.*

In general, if the given sum be $a + b$, we have, by the definition of subtraction,

$$(a + b) - b = a \quad \text{(i.)}, \qquad \text{and} \qquad (a + b) - a = b \quad \text{(ii.)}.$$

2. The minuend is, as a rule, a single number, and does not appear as a sum of two numbers, one of which is the given subtrahend. We must, therefore, derive from the definition of subtraction a principle which will enable us to subtract any one number from any other.

Ex. 1. Subtract ${}^{+}5$ from ${}^{+}7$.

In ${}^{+}7 - {}^{+}5$, the minuend, ${}^{+}7$, is to be expressed as the sum of two numbers, *one of which is* ${}^{+}5$. But

$$\begin{aligned} {}^{+}7 &= {}^{+}7 + ({}^{-}5 + {}^{+}5), \text{ by § 1, Art. 13,} \\ &= ({}^{+}7 + {}^{-}5) + {}^{+}5, \text{ by Assoc. Law.} \end{aligned}$$

Therefore, by definition of subtraction,

$$\begin{aligned} {}^{+}7 - {}^{+}5 &= [({}^{+}7 + {}^{-}5) + {}^{+}5] - {}^{+}5 \\ &= {}^{+}7 + {}^{-}5 = {}^{+}2. \end{aligned}$$

That is, *to subtract* ${}^{+}5$ *from* ${}^{+}7$ *is equivalent to adding* ${}^{-}5$ *to* ${}^{+}7$.

Ex. 2. Subtract ${}^{-}5$ from ${}^{+}7$.

We have

$$\begin{aligned} {}^{+}7 - {}^{-}5 &= [({}^{+}7 + {}^{+}5) + {}^{-}5] - {}^{-}5 \\ &= {}^{+}7 + {}^{+}5 = {}^{+}12. \end{aligned}$$

That is, *to subtract* $^-5$ *from* $^+7$ *is equivalent to adding* $^+5$ *to* $^+7$.

These examples illustrate the following principle:

To subtract one number from another number, reverse the sign of quality of the subtrahend, and add.

Or, stated symbolically,

$$N - {}^+b = N + {}^-b \text{ (i.)}, \qquad N - {}^-b = N + {}^+b \text{ (ii.)}.$$

E.g., $^+2 - {}^+3 = {}^+2 + {}^-3, = {}^-1.$ $\quad {}^-2 - {}^+3 = {}^-2 + {}^-3, = {}^-5.$

$^+2 - {}^-3 = {}^+2 + {}^+3, = {}^+5.$ $\quad {}^-2 - {}^-3 = {}^-2 + {}^+3, = {}^+1.$

3. The proof of the principle enunciated in Art. 2, is as follows:

Let N denote any number, *positive or negative*. Then in

$$N - {}^+b,$$

N is to be expressed as the sum of two numbers, *one of which is* ^+b.

We have $\quad N = N + ({}^-b + {}^+b)$, by § 1, Art. 13,

$= (N + {}^-b) + {}^+b$, by Assoc. Law.

Therefore, by the definition of subtraction,

$$N - {}^+b = [(N + {}^-b) + {}^+b] - {}^+b = N + {}^-b.$$

In like manner $\quad N - {}^-b = N + {}^+b.$

Successive Additions and Subtractions.

4. Successive subtractions are carried out by applying the principle of subtraction at each step of the process.

E.g., $^+7 - {}^-2 - {}^+4 - {}^-8 = {}^+9 - {}^+4 - {}^-8 = {}^+5 - {}^-8 = {}^+13.$

In like manner successive additions and subtractions are performed.

E.g., $^+9 + {}^-5 - {}^+2 + {}^+4 = {}^+4 - {}^+2 + {}^+4 = {}^+2 + {}^+4 = {}^+6.$

5. The following definition is based upon the principle that every operation of subtraction is equivalent to an operation of addition.

An **Algebraic Sum** is an expression which consists of a chain of indicated additions and subtractions.

E.g., $a - b$, $x + y - z$, etc., are algebraic sums.

EXERCISES III.

Find the results of the following indicated subtractions:

1. $({}^+2 + {}^+9) - {}^+2.$ **2.** $({}^+4 + {}^-5) - {}^-5.$ **3.** $({}^-7 + {}^-11) - {}^-11.$

4. $({}^-x + {}^-7) - {}^-7.$ **5.** $({}^-m + {}^+n) - {}^-m.$ **6.** $({}^-m + {}^+n) - {}^+n.$

7. $^{+}18 - {}^{+}5$. 8. $^{-}28 - {}^{+}17$. 9. $^{+}7 - {}^{+}8$. 10. $^{+}11 - {}^{+}25$

11. $^{+}41 - {}^{-}50$. 12. $^{-}30 - {}^{-}35$. 13. $^{-}65 - {}^{-}45$. 14. $^{+}44 - {}^{-}52$

15. $^{-}2 + {}^{+}3 - {}^{+}4$. 16. $^{+}11 - {}^{+}13 + {}^{-}12$. 17. $^{-}7 - {}^{-}11 + {}^{-}2$.

18. $({}^{-}2 - {}^{-}5) + ({}^{-}6 + {}^{+}3)$. 19. $({}^{+}11 - {}^{-}8) - ({}^{-}2 - {}^{+}3)$.

Subtract:

20. $^{+}3$ from ^{+}a, when $a > 3$. 21. $^{+}3$ from ^{+}a, when $a < 3$.

22. $^{-}17$ from ^{-}x, when $x > 17$. 23. $^{-}17$ from ^{-}x, when $x < 17$.

24. $^{-}11$ from ^{+}x. 25. $^{+}14$ from ^{-}y.

What is the value of $a - b$,

26. When $a = {}^{+}4$, $b = {}^{+}3$? 27. When $a = {}^{+}5$, $b = {}^{+}6$?

28. When $a = {}^{-}7$, $b = {}^{+}8$? 29. When $a = {}^{-}4$, $b = {}^{-}9$?

30. When $a = {}^{-}2 + {}^{-}3$, $b = {}^{+}11 - {}^{+}4$?

31. When $a = {}^{+}5 - {}^{-}4$, $b = {}^{-}18 - {}^{+}7$?

What is the value of $a - b$, wherein $a = m + n$ and $b = p - q$,

32. When $m = {}^{-}1$, $n = {}^{+}2$, $p = {}^{-}3$, $q = {}^{+}4$?

33. When $m = {}^{+}5$, $n = {}^{-}6$, $p = {}^{-}11$, $q = {}^{-}12$?

6. The following examples illustrate the meaning of results in the subtraction of algebraic numbers.

Ex. 1. Two men, A and B, starting from the same point, P, walk at different rates in the same direction, A 8 miles to the point Q, B 11 miles to the point R. How far is B then from A?

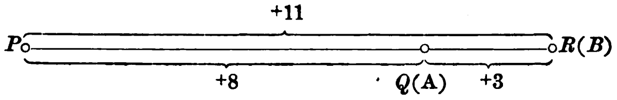

FIG. 1.

As we have seen in Ch. I., distances in one and the same direction may be represented by numbers of the same sign. Let distances toward the right be taken positively, as in Fig. 1, and consequently distances toward the left negatively.

The distance of B from A is then represented by QR, and

$$QR = PR - PQ = {}^{+}11 - {}^{+}8 = {}^{+}3.$$

The *positive* result shows that B is 3 miles to the *right* of A.

In general, however far either may walk, the distance of B from A will always be obtained by subtracting A's distance from the starting point from B's distance from the same point.

Ex. 2. If A walks 8 miles to the left and B 11 miles to the left, their distances from P are both negative, as in Fig. 2.

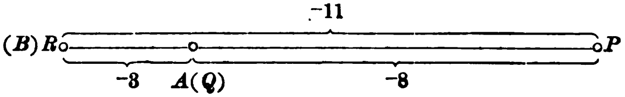

FIG. 2.

We then have $QR = PR - PQ = {}^{-}11 - {}^{-}8 = {}^{-}3.$

The *negative* result shows that B is 3 miles to the *left* of A.

In a similar way other variations of the problem may be interpreted.

EXERCISES IV.

1. A's assets are a dollars and B's are b dollars. What number expresses the excess of A's assets over B's, if assets be taken positively? What number, if assets be taken negatively?

What are the meanings of the results of Ex. 1,

2. When $a = 3500$, $b = 2750$? **3.** When $a = 2000$, $b = 2000$?

4. When $a = 2600$, $b = 3000$?

5. The temperature in Chicago on a certain day was $a°$ and in Philadelphia $b°$. What number expresses the excess of temperature in Chicago over that in Philadelphia?

What is the meaning of the result of Ex. 5, taking temperature above zero positively,

6. When $a = {}^{+}90$, $b = {}^{+}68$? **7.** When $a = {}^{+}65$, $b = {}^{+}98$?

8. When $a = {}^{-}12$, $b = {}^{-}4$? **9.** When $a = {}^{-}5$, $b = {}^{-}8$?

10. When $a = {}^{+}5$, $b = {}^{-}2$? **11.** When $a = {}^{-}6$, $b = {}^{+}3$?

The Associative and Commutative Laws for Subtraction.

7. If a number, preceded by the sign $+$, or the sign $-$, stand first in a chain of additions and subtractions, or first within parentheses, it may be regarded as added to, or subtracted from, 0. Thus,

$$+{}^{+}8 - {}^{+}3 = 0 + {}^{+}8 - {}^{+}3, \quad -{}^{+}3 + {}^{+}8 = 0 - {}^{+}3 + {}^{+}8.$$

Since every operation of subtraction is equivalent to an operation of addition, it follows that the Associative and Commutative Laws which were proved for addition hold also for subtraction, and for successive additions and subtractions.

Ex. $+{}^+8 - {}^+3 = +{}^+8 + {}^-3$, since $-{}^+3 = +{}^-3$
$= +{}^-3 + {}^+8$, by Comm. Law
$= -{}^+3 + {}^+8$, since $+{}^-3 = -{}^+3$.

Observe that *in changing the order of the operations the sign of operation, + or −, must be transferred with each number.*

The method of applying the Associative Law depends upon a proper use of parentheses, which will be taken up in the next article.

EXERCISES V.

Find in three different ways, by applying the Commutative Law, the values of:

1. $^+8 + {}^-3 - {}^+4$.
2. $^-17 - {}^+12 + {}^-5$.
3. $^+28 - {}^-14 + {}^-2$.
4. $^-31 - {}^-17 + {}^-36 + {}^+46 - {}^-11 - {}^+19 + {}^-49 + {}^+11$.
5. $^-45 + {}^+31 - {}^-15 - {}^+12 + {}^-5 - {}^-9 + {}^-8 + {}^+4$.

Find, in the most convenient way, the values of:

6. $^+103 - {}^-12 - {}^+3$.
7. $^-799 - {}^-11 + {}^-1$.

Removal of Parentheses.

8. We have $^+9 + ({}^+5 + {}^+6) = {}^+9 + {}^+5 + {}^+6$,
since to add the sum $^+5 + {}^+6$ is equivalent to adding successively the single numbers of that sum.

Again, $^+9 + ({}^+5 - {}^+6) = {}^+9 + ({}^+5 + {}^-6)$, since $-{}^+6 = +{}^-6$,
$= {}^+9 + {}^+5 + {}^-6$, removing parentheses,
$= {}^+9 + {}^+5 - {}^+6$, since $+{}^-6 = -{}^+6$.

This example illustrates the following principle:

(i.) *When the sign of addition, +, precedes parentheses, they may be removed, and the signs of operation, + and −, within them be left unchanged;* that is,

$$\boldsymbol{N} + (+\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{N} + \boldsymbol{a} + \boldsymbol{b},$$
$$\boldsymbol{N} + (+\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{N} + \boldsymbol{a} - \boldsymbol{b}, \text{ etc.}$$

We have $^+9 - ({}^+5 + {}^+6) = {}^+9 - {}^+5 - {}^+6$,
since to subtract the sum $^+5 + {}^+6$ is equivalent to subtracting successively the single numbers of that sum.

Again, $^+9 - ({}^+5 - {}^+6) = {}^+9 - ({}^+5 + {}^-6)$, since $-{}^+6 = +{}^-6$,
$= {}^+9 - {}^+5 - {}^-6$, removing parentheses,
$= {}^+9 - {}^+5 + {}^+6$, since $-{}^-6 = +{}^+6$.

This example illustrates the following principle:

(ii.) *When the sign of subtraction, $-$, precedes parentheses, they may be removed, if the signs of operation within them be reversed from $+$ to $-$, and from $-$ to $+$;* that is,

$$\boldsymbol{N} - (+ \boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{N} - \boldsymbol{a} - \boldsymbol{b},$$
$$\boldsymbol{N} - (+ \boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{N} - \boldsymbol{a} + \boldsymbol{b}, \text{ etc.}$$

For, $N + ({}^{+}a + {}^{+}b) = N + {}^{+}a + {}^{+}b,$

since to add the sum ${}^{+}a + {}^{+}b$ is equivalent to adding successively the single numbers of that sum.

$$\begin{aligned} N + ({}^{+}a - {}^{+}b) &= N + ({}^{+}a + {}^{-}b), \text{ since } - {}^{+}b = + {}^{-}b, \\ &= N + {}^{+}a + {}^{-}b, \text{ removing parentheses,} \\ &= N + {}^{+}a - {}^{+}b, \text{ since } + {}^{-}b = - {}^{+}b. \end{aligned}$$

Evidently the preceding proof does not depend upon the signs of quality of the numbers within the parentheses, nor upon how many numbers are inclosed. In a similar manner (ii.) is proved.

Insertion of Parentheses.

9. The insertion of parentheses is the converse of the process of removing them.

(i.) *An expression may be inclosed within parentheses preceded by the sign of addition, if the signs of operation, $+$ and $-$, preceding the numbers inclosed within the parentheses remain unchanged.*

E.g.,
$$\begin{aligned} + {}^{+}7 - {}^{+}5 + {}^{-}3 - {}^{-}4 &= + {}^{+}7 + (- {}^{+}5 + {}^{-}3 - {}^{-}4) \\ &= + {}^{+}7 - {}^{+}5 + (+ {}^{-}3 - {}^{-}4) \\ &= + {}^{+}7 - {}^{+}5 + {}^{-}3 + (- {}^{-}4). \end{aligned}$$

(ii.) *An expression may be inclosed within parentheses preceded by the sign of subtraction, if the signs of operation preceding the numbers inclosed within the parentheses be reversed, from $+$ to $-$ and from $-$ to $+$.*

E.g.,
$$\begin{aligned} + {}^{+}7 - {}^{+}5 + {}^{-}3 - {}^{-}4 &= + {}^{+}7 - (+ {}^{+}5 - {}^{-}3 + {}^{-}4) \\ &= + {}^{+}7 - {}^{+}5 - (- {}^{-}3 + {}^{-}4) \\ &= + {}^{+}7 - {}^{+}5 + {}^{-}3 - (+ {}^{-}4). \end{aligned}$$

The insertion of parentheses is a direct application of the Associative Law.

EXERCISES VI.

Find the values of the following expressions, first removing parentheses:

1. ${}^{+}12 + ({}^{+}4 + {}^{-}6)$. 2. ${}^{-}15 + ({}^{-}6 - {}^{+}2)$.
3. ${}^{+}28 + ({}^{-}5 - {}^{+}6)$. 4. ${}^{+}18 + ({}^{-}2 + {}^{-}3 - {}^{-}5)$.
5. ${}^{+}11 - ({}^{+}12 + {}^{-}5)$. 6. ${}^{+}15 - ({}^{-}6 - {}^{+}2)$.
7. ${}^{-}17 - ({}^{-}3 - {}^{-}5)$. 8. ${}^{-}21 - ({}^{-}4 + {}^{-}5 - {}^{-}6)$.
9. $m + (n - p)$, when $m = {}^{-}4$, $n = {}^{-}6$, $p = {}^{+}5$.
10. $x - (y - z)$, when $x = {}^{+}3$, $y = {}^{-}4$, $z = {}^{+}5$.

Insert parentheses in the expression ${}^{+}8 - {}^{-}5 + {}^{-}3 - {}^{+}7$,

11. To inclose the last three numbers preceded by the sign $+$; preceded by the sign $-$.

12. To inclose the last two numbers preceded by the sign $+$; preceded by the sign $-$.

13. To inclose the first and third numbers preceded by the sign $+$; preceded by the sign $-$.

10. In ordinary Arithmetic, to subtract a number from any number decreases the latter.

E.g., $7 - 4 = 3$, and $3 < 7$.

But such is not always the case in subtracting one algebraic number from another. Thus,

$$ {}^{+}7 - {}^{+}4 = {}^{+}3, \text{ and } {}^{+}3 < {}^{+}7; $$

but

$$ {}^{+}7 - {}^{-}4 = {}^{+}11, \text{ and } {}^{+}11 > {}^{+}7. $$

Property of Zero in Subtraction.

11. From § 1, Art. 13, we have $N + 0 = N$.

If, therefore, from N, which is the sum of N and 0, be subtracted either N or 0, the remainder is 0 or N, respectively, by the definition of subtraction.

That is,

$$ \boldsymbol{N} - \boldsymbol{N} = \mathbf{0}, \text{ and } \boldsymbol{N} - \mathbf{0} = \boldsymbol{N}. \qquad \text{(i.)} $$

§ 3. MULTIPLICATION OF ALGEBRAIC NUMBERS.

1. As in Arithmetic, the number multiplied is called the **Multiplicand,** the number that multiplies the **Multiplier,** and the result the **Product.** In ordinary Arithmetic, multiplication by an integer is defined as an abbreviated addition. Thus, to multiply 4 by 3, the number 4 is used three times as a summand; or

$$4 \times 3 = 4 + 4 + 4.$$

Now the number 3 stands for an aggregate of three units; or

$$3 = 1 + 1 + 1.$$

We thus see that, just as 3 is obtained by taking the unit, 1, three times as a summand, so the product 4×3 is obtained by taking 4 three times as a summand.

2. We are thus naturally led to the following definition of multiplication:

The product is obtained from the multiplicand just as the multiplier is obtained from the positive unit.

The above definition is an extension of the meaning of arithmetical multiplication when the multiplier is an integer, and gives an intelligible meaning to arithmetical multiplication when the multiplier is a fraction.

Thus, $\frac{2}{3}$ is obtained from the unit, 1, by taking one-third of the latter twice as a summand; or

$$\tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3}.$$

In like manner, to multiply 5 by $\frac{2}{3}$, we take one-third of 5 twice as a summand; or

$$5 \times \tfrac{2}{3} = \tfrac{5}{3} + \tfrac{5}{3} = \tfrac{10}{3}.$$

3. There are two cases to be considered in the multiplication of algebraic numbers.

(i.) **The Multiplier Positive.**—Ex. **1.** Multiply $^{+}4$ by $^{+}3$.

By the definition of multiplication, the product,

$$^{+}4 \times {}^{+}3,$$

is obtained from $^{+}4$ just as $^{+}3$ is obtained from the positive unit. But $^{+}3$ is obtained from the positive unit by taking the latter *three times as a summand;* or

$$^{+}3 = {}^{+}1 + {}^{+}1 + {}^{+}1.$$

Consequently the required product is obtained by taking $^{+}4$ *three times as a summand;* or

$$^{+}4 \times {}^{+}3 = {}^{+}4 + {}^{+}4 + {}^{+}4 = {}^{+}(4 + 4 + 4) = {}^{+}(4 \times 3) = {}^{+}12.$$

Ex. 2. Multiply $^{-}4$ by $^{+}3$.

By the definition of multiplication, we have

$$^{-}4 \times {}^{+}3 = {}^{-}4 + {}^{-}4 + {}^{-}4 = {}^{-}(4+4+4) = {}^{-}(4 \times 3) = {}^{-}12.$$

(ii.) **The Multiplier Negative.** — Ex. 3. Multiply $^{+}4$ by $^{-}3$.

By the definition of multiplication, the product,

$$^{+}4 \times {}^{-}3,$$

is obtained from $^{+}4$ just as $^{-}3$ is obtained from the positive unit. But

$$^{-}3 = {}^{-}1 + {}^{-}1 + {}^{-}1 = - {}^{+}1 - {}^{+}1 - {}^{+}1;$$

that is, $^{-}3$ is obtained *by subtracting the positive unit,* $^{+}1$, *three times in succession from* 0. Consequently, the required product is obtained *by subtracting the multiplicand,* $^{+}4$, *three times in succession from* 0; or

$$^{+}4 \times {}^{-}3 = - {}^{+}4 - {}^{+}4 - {}^{+}4 = + {}^{-}4 + {}^{-}4 + {}^{-}4 = {}^{-}(4 \times 3).$$

Ex. 4. Multiply $^{-}4$ by $^{-}3$.

By the definition of multiplication, we have

$$^{-}4 \times {}^{-}3 = - {}^{-}4 - {}^{-}4 - {}^{-}4 = + {}^{+}4 + {}^{+}4 + {}^{+}4 = {}^{+}(4 \times 3).$$

4. In Art. 3 the examples were limited to the multiplication of integers having the same or opposite signs.

But the essential part of the results therein obtained is the sign of the product.

Since this sign depends only upon the signs of the multiplicand and multiplier, and not upon their absolute values, the sign of the product in each example would have been the same as above, if the multiplicand and multiplier, either or both, had been fractions.

These examples illustrate the following **Rule of Signs for Multiplication:**

The product of two numbers having like signs is positive; and the product of two numbers having unlike signs is negative. Or, stated symbolically,

$$^{+}a \times {}^{+}b = {}^{+}(ab), \qquad {}^{-}a \times {}^{+}b = {}^{-}(ab),$$
$$^{-}a \times {}^{-}b = {}^{+}(ab). \qquad {}^{+}a \times {}^{-}b = {}^{-}(ab).$$

5. The proof of the principle enunciated in Art. 4 is as follows:

The product $$^+a \times {}^+b,$$

wherein a and b *are, as yet, limited to integral values*, is obtained from ^+a just as ^+b is obtained from the positive unit. But ^+b is obtained from $^+1$ by taking the latter b *times as a summand;* or,

$$^+b = {}^+1 + {}^+1 + {}^+1 + \cdots b \text{ summands.}$$

Therefore the required product is obtained by taking ^+a *as a summand* b *times;* or

$$\begin{aligned} ^+a \times {}^+b &= {}^+a + {}^+a + {}^+a + \cdots b \text{ summands} \\ &= {}^+(a + a + a + \cdots b \text{ summands}) = {}^+(ab). \end{aligned}$$

Consequently, $^+a \times {}^+b = {}^+(ab)$.

In like manner, the other products are proved.

Since the essential part of the above proof is the sign of the product, the results hold when a and b have fractional values.

Continued Products.

6. The results of the preceding articles may be applied to determine the value of a chain of indicated multiplications, *i.e.*, of a *continued product.*

E.g.,

$$\begin{aligned} ^+a \times {}^+b \times {}^+c &= {}^+(ab) \times {}^+c = {}^+(abc), \\ ^+a \times {}^+b \times {}^-c &= {}^+(ab) \times {}^-c = {}^-(abc), \\ ^+a \times {}^-b \times {}^-c &= {}^-(ab) \times {}^-c = {}^+(abc), \\ ^-a \times {}^-b \times {}^-c &= {}^+(ab) \times {}^-c = {}^-(abc). \end{aligned}$$

These equations illustrate a more general rule of signs:

A continued product which contains no negative number, or an even number of negative numbers, is positive; one that contains an odd number of negative numbers is negative.

In practice the sign of a required product may first be determined by inspection, and that sign prefixed to the product of the absolute values of the numbers in the continued product.

E.g., the sign of the product

$$(^+2) \times (^-3) \times (^-7) \times (^+4) \times (^-5)$$

is *negative*, since it contains *three* negative numbers; the product of the absolute values is 840. Consequently,

$$(^+2) \times (^-3) \times (^-7) \times (^+4) \times (^-5) = {}^-840.$$

EXERCISES VII.

Find the values of the following indicated multiplications:

1. $^+2 \times {}^+3$.
2. $^+5 \times {}^+7$.
3. $^-3 \times {}^+7$.
4. $^-11 \times {}^+9$.
5. $^+5 \times {}^-6$.
6. $^+7 \times {}^-4$.
7. $^-7 \times {}^-9$.
8. $^-22 \times {}^-6$.

Find the values of:

9. $({}^+15 \times {}^-4) + ({}^-12 \times {}^+1)$.
10. $({}^+16 \times {}^-3) - ({}^+5 \times {}^+7)$.
11. $({}^-1 \times {}^+11) - ({}^-22 \times {}^+2)$.
12. $({}^-52 \times {}^-2) - ({}^+12 \times {}^-3)$.

Find the values of the following continued products:

13. $^-2 \times {}^+4 \times {}^-3$.
14. $^-5 \times {}^-6 \times {}^+7$.
15. $^+12 \times {}^-2 \times {}^-5 \times {}^-4$.

What is the value of $(a - b)(c + d)$,

16. When $a = {}^-2,\ b = {}^+4,\ c = {}^-5,\ d = {}^+6$?
17. When $a = {}^+5,\ b = {}^-8,\ c = {}^-9,\ d = {}^+10$?

What is the value of $abcd$,

18. When $a = {}^-2,\ b = {}^+3,\ c = {}^-4,\ d = {}^+5$?
19. When $a = {}^-7,\ b = {}^+2,\ c = {}^-3,\ d = {}^-5$?

The Commutative Law for Multiplication.

7. In an indicated multiplication, the number which follows the symbol of multiplication is the multiplier. Thus, in

$$^-4 \times {}^+3 = {}^-4 + {}^-4 + {}^-4, = {}^-12,$$

the multiplier is $^+3$; while in

$$^+3 \times {}^-4 = - {}^+3 - {}^+3 - {}^+3 - {}^+3, = {}^-12,$$

the multiplier is $^-4$. But the result is the same, whichever of the two numbers, $^+3$ or $^-4$, is used as the multiplier.

This example illustrates the following principle:

The Commutative Law. — *The product of two numbers is the same, if either be taken as the multiplier and the other as the multiplicand;* or, stated symbolically,

$$a \times b = b \times a.$$

8. It follows from the rule of signs, Art. 4, that the signs of the multiplier and multiplicand may be interchanged without affecting the sign of the product.

E.g., $^+3 \times {}^-4 = {}^-(3 \times 4) = {}^-12$, and $^-3 \times {}^+4 = {}^-(3 \times 4) = {}^-12$.

This principle is called the *Commutative Law of Signs.*

We have, therefore, to prove the Commutative Law only for the multiplication of absolute numbers.

Take first a particular example: $4 \times 3 = 3 \times 4$.

Consider the following arrangement of units in rows and columns:

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

The total number of units in this arrangement is obtained either by multiplying the number in each row, 4, by the number of rows, 3, giving 4×3; or by multiplying the number of units in each column, 3, by the number of columns, 4, giving 3×4.

Consequently, $$4 \times 3 = 3 \times 4.$$

In general, consider the following arrangement of units in rows and columns:

$$\begin{array}{rl} & \overline{a \text{ units in each row}} \\ b \text{ rows} & \left\{\begin{array}{cccccccc} 1 & 1 & 1 & 1 & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}\right. \end{array}$$

The total number of units in this arrangement is obtained either by multiplying the number in each row, a, by the number of rows, b, giving $a \times b$; or by multiplying the number in each column, b, by the number of columns, a, giving $b \times a$.

Consequently, $$a \times b = b \times a,$$

wherein a and b are *absolute integers*.

Consider next the case in which a and b denote absolute fractions.

$$\begin{aligned} \frac{4}{5} \times \frac{2}{3} &= \frac{4}{5 \times 3} + \frac{4}{5 \times 3}, \text{ by the definition of multiplication,} \\ &= \frac{4+4}{5 \times 3} = \frac{4 \times 2}{5 \times 3} \\ &= \frac{2 \times 4}{3 \times 5}, \text{ since } 4 \times 2 = 2 \times 4 \text{ and } 5 \times 3 = 3 \times 5, \\ &= \frac{2+2+2+2}{3 \times 5} = \frac{2}{3 \times 5} + \frac{2}{3 \times 5} + \frac{2}{3 \times 5} + \frac{2}{3 \times 5} = \frac{2}{3} \times \frac{4}{5}. \end{aligned}$$

Similar reasoning can be applied to the product of any two absolute fractions.

Consequently, the Commutative Law for multiplication,

$$a \times b = b \times a,$$

holds for all values of a and b, positive or negative, integral or fractional.

The Associative Law for Multiplication.

9. In finding the value of a continued product in Art. 6, the indicated operations were performed successively from left to right.

E.g., $(^+4 \times {}^+3) \times {}^-2 = {}^+12 \times {}^-2 = {}^-24.$

But the same result is obtained if $^+3$ be first multiplied by $^-2$ and then $^+4$ be multiplied by the product.

E.g., $^+4 \times (^+3 \times {}^-2) = {}^+4 \times {}^-6 = {}^-24.$

This example illustrates the following principle:

The Associative Law. — *The product of three numbers is the same in whichever way two successive numbers are grouped or associated in the process of multiplying;* or, stated symbolically,

$$(ab)c = a(bc).$$

10. For the reason stated in Art. 8, it is sufficient to prove this law for absolute numbers.

Consider the following arrangement of b's in rows and columns:

$$\begin{array}{rc} & c \text{ columns of } b\text{'s} \\ a \text{ rows of } b\text{'s} & \left\{\begin{array}{cccccccc} b & b & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right. \end{array}$$

Each row contains $b \times c$ units; and, since there are a rows, the total number of units is

$$(b \times c) \times a, \text{ or } a \times (b \times c), \text{ by Art. 7.}$$

But each column contains $b \times a$, or $a \times b$ units; and, since there are c columns, the total number of units is

$$(a \times b) \times c.$$

Therefore $(a \times b) \times c = a \times (b \times c).$

In the above representation, the values of a, b, and c are limited to absolute integers. It can be shown, however, as in Art. 8, that the law holds also for absolute fractional values of a, b, c.

E.g., $\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}=\frac{2\times 4}{3\times 5}\times\frac{6}{7}=\frac{2\times 4\times 6}{3\times 5\times 7}=\frac{2\times(4\times 6)}{3\times(5\times 7)}=\frac{2}{3}\times\left(\frac{4}{5}\times\frac{6}{7}\right)$.

We conclude, therefore, that the Associative Law for multiplication holds for all values of a, b, c, *positive or negative*, *integral or fractional.*

11. The Associative and Commutative Laws may be extended as follows:

The value of a product of three or more numbers remains the same if, in performing the indicated multiplications, the order of the numbers be changed, or if two or more numbers be associated in any way.

E.g., $^+2\times{}^-3\times{}^-4={}^+2\times{}^-4\times{}^-3={}^-4\times{}^+2\times{}^-3=\text{etc.}$

$^+2\times{}^-3\times{}^-4\times{}^+5={}^+2\times({}^-3\times{}^-4)\times{}^+5={}^+2\times{}^-3\times({}^-4\times{}^+5)=\text{etc.}$

In general,

$$abc=acb=bca=bac=cab=cba.$$

$$abcd=a(bcd)=a(bc)d=ab(cd).$$

The two laws may be applied simultaneously.

E.g.,

$^+2\times{}^-3\times{}^-4\times{}^+5={}^+2\times({}^+5\times{}^-3)\times{}^-4={}^-3\times({}^-4\times{}^+2\times{}^+5)=\text{etc.}$

In general, $abcd=a(cb)d=c(adb)=\text{etc.}$

12. Since the multiplier and the multiplicand can be interchanged without affecting the value of the product, they are both given the common name **Factor**.

Thus, a and b are the factors of the product ab.

For a similar reason, each number in a continued product is called a factor of the product.

Thus a, b, c, and d are factors of $abcd$.

In subsequent work we shall, for convenience in writing, frequently place the multiplier on the left.

EXERCISES VIII.

1. Express the sum, $^+4+{}^+4+{}^+4$, as a sum of summands each equal to $^+3$.

2. Express the sum, $^-5+{}^-5+{}^-5+{}^-5$, as a sum of summands each equal to $^-4$.

3. Express the sum, $^-(\frac{3}{4}) + {}^-(\frac{3}{4}) + {}^-(\frac{3}{4}) + {}^-(\frac{3}{4}) + {}^-(\frac{3}{4}) + {}^-(\frac{3}{4})$, as a sum of summands each equal to $^-(\frac{6}{4})$.

4. Express the sum, $^+9 + {}^+9 + {}^+9 + \cdots a$ summands, as a sum of summands each equal to ^+a.

5. Express the sum, $^-5 + {}^-5 + \cdots x$ summands, as a sum of summands each equal to ^-x.

Find, in the most convenient way, the values of:

6. $^-17 \times {}^+5 \times {}^-2$. **7.** $^+38 \times {}^-2\frac{1}{2} \times {}^-4$. **8.** $^-139 \times {}^-3 \times {}^+33\frac{1}{3}$.

9. $^+228 \times {}^+250 \times {}^-4$. **10.** $^-139 \times {}^-8 \times {}^+12\frac{1}{2}$. **11.** $^-17 \times {}^-16\frac{2}{3} \times {}^-3$.

Find, in the most convenient way, the value of *abc*:

12. When $a = {}^-4$, $b = {}^+33\frac{1}{3}$, $c = {}^-9$. **13.** When $a = {}^+19$, $b = {}^-66\frac{2}{3}$, $c = {}^-3$.

Find, in the most convenient way, the value of *abcd*,

14. When $a = {}^-37\frac{1}{2}$, $b = {}^+5$, $c = {}^-3$, $d = {}^+8$.

15. When $a = {}^-12\frac{1}{2}$, $b = {}^-16$, $c = {}^+33\frac{1}{3}$, $d = {}^-6$.

§ 4. DIVISION OF ALGEBRAIC NUMBERS.

1. **Division** *is the inverse of multiplication.* In multiplication two factors are given, and it is required to find their product. In division the product of two factors and one of them are given, and it is required to find the other factor. As in ordinary Arithmetic, the given product is called the **Dividend,** the given factor the **Divisor,** and the required factor the **Quotient.**

E.g., Since $^-28 = {}^-4 \times {}^+7$,

therefore, $^-28 \div {}^+7 = {}^-4$, and $^-28 \div {}^-4 = {}^+7$.

2. From the definition of division we infer the following principle:

If the product of two factors be divided by either of the factors, the quotient is the other factor.

In general, if the given product be $a \times b$, we have, by the definition of division,

$$(a \times b) \div b = a, \text{ and } (a \times b) \div a = b. \qquad \text{(i.)}$$

3. The dividend is, as a rule, a single number and does not appear as the product of two factors, one of which is the divisor.

Since the absolute value of the product of two factors is equal to the product of their absolute values, it follows, from the definition of division, that the absolute value of the quotient is equal to the quotient of the absolute values of the dividend and the divisor.

By the definition of division, the equations of § 3, Art. 4, may be written

$$^{+}(ab) \div {}^{+}a = {}^{+}b;\quad {}^{-}(ab) \div {}^{-}a = {}^{+}b;$$
$$^{+}(ab) \div {}^{-}a = {}^{-}b;\quad {}^{-}(ab) \div {}^{+}a = {}^{-}b.$$

From these equations, we derive the following **Rule of Signs for Division:**

Like signs of dividend and divisor give a positive quotient; unlike signs of dividend and divisor give a negative quotient.

E.g.,
$$^{+}8 \div {}^{+}2 = {}^{+}4;\quad {}^{-}8 \div {}^{-}2 = {}^{+}4;$$
$$^{-}8 \div {}^{+}2 = {}^{-}4;\quad {}^{+}8 \div {}^{-}2 = {}^{-}4.$$

4. In a chain of indicated divisions, the operations are to be performed successively from left to right.

E.g.,
$$^{-}16 \div {}^{+}4 \div {}^{-}2 = {}^{-}4 \div {}^{-}2 = {}^{+}2;$$
$$^{+}210 \div {}^{-}3 \div {}^{-}2 \div {}^{+}5 = {}^{-}70 \div {}^{-}2 \div {}^{+}5 = {}^{+}35 \div {}^{+}5 = {}^{+}7.$$

Likewise, in a chain of indicated multiplications and divisions, the operations are to be performed successively from left to right.

E.g.,
$$^{-}375 \times {}^{+}3 \div {}^{-}5 \times {}^{+}2 \div {}^{-}9 = {}^{-}1125 \div {}^{-}5 \times {}^{+}2 \div {}^{-}9$$
$$= {}^{+}225 \times {}^{+}2 \div {}^{-}9 = {}^{+}450 \div {}^{-}9 = {}^{-}50.$$

5. In a succession of additions, subtractions, multiplications, and divisions, the multiplications and divisions are first to be performed, and then the additions and subtractions.

E.g.,
$$^{+}2 \times {}^{-}3 + {}^{-}4 \times {}^{+}5 = {}^{-}6 + {}^{-}20 = {}^{-}26.$$

When a different order of performing the operations is proposed, the required order must be indicated by the insertion of parentheses.

E.g.,
$$^{+}2 \times ({}^{-}3 + {}^{-}4) \times {}^{+}5 = {}^{+}2 \times {}^{-}7 \times {}^{+}5 = {}^{-}70,$$

not $^{-}26$, as before.

6. From the definition of division, we have

$$\textit{Quotient} \times \textit{Divisor} = \textit{Dividend}.$$

Since the quotient of a divided by b is $a \div b$, we have

$$(a \div b) \times b = a, \text{ or } a \div b \times b = a. \qquad (1)$$

From (1) and Art. 2 (i.), we derive the following principle:

If a given number be first divided and then multiplied by one and the same number, or be first multiplied and then divided by one and the same number, the result is the given number; or, stated symbolically,

$$N \times b \div b = N \div b \times b = N \times {}^{+}1 = N \div {}^{+}1 = N.$$

That is, $$\times\, b \div b = \div\, b \times b = \times\, {}^{+}1 = \div\, {}^{+}1, \qquad (2)$$

whatever number be placed on the left of the two indicated operations.

E.g., $^{-}11 \times {}^{+}3 \div {}^{+}3 = {}^{-}11, \quad {}^{+}11 \div {}^{-}5 \times {}^{-}5 = {}^{+}11.$

EXERCISES IX.

Find the values of the following indicated divisions:

1. $(^{-}27 \times {}^{+}3) \div {}^{+}3.$ **2.** $(^{-}27 \times {}^{+}3) \div {}^{-}27.$ **3.** $(^{+}2\frac{2}{3} \times {}^{+}5\frac{3}{8}) \div {}^{+}2\frac{2}{3}.$

4. $^{+}27 \div {}^{+}9.$ **5.** $^{+}81 \div {}^{-}9.$ **6.** $^{-}33 \div {}^{+}11.$ **7.** $^{-}105 \div {}^{-}7.$

Find the values of:

8. $(^{-}16 \div {}^{+}2) + (^{+}18 \div {}^{-}3).$ **9.** $(^{-}24 \div {}^{-}8) - (^{-}36 \div {}^{+}6).$

10. $(^{-}15 \div {}^{-}5) - (^{+}100 \div {}^{-}25) + (^{-}200 \div {}^{+}8).$

Find the values of:

11. $^{+}210 \div {}^{-}5 \div {}^{-}7 \div {}^{+}2.$ **12.** $^{+}375 \div {}^{-}5 \times {}^{+}2 \div {}^{-}3.$

13. $^{-}280 \div {}^{-}4 \times {}^{+}2 \times {}^{-}3 \div {}^{-}42.$ **14.** $^{+}15 \times {}^{-}6 \div {}^{-}2 \div {}^{+}5 \times {}^{+}4.$

Find the values of the following expressions:

15. $^{+}180 \div {}^{-}36 + {}^{-}4 \times {}^{-}2.$ **16.** $^{-}25 \times {}^{+}4 - {}^{+}36 \div {}^{-}12.$

17. $^{+}48 \times {}^{-}2 - {}^{-}96 \div {}^{-}24.$ **18.** $^{-}7 + {}^{+}15 \div {}^{+}3 \times {}^{-}2 - {}^{-}28 \div {}^{+}4.$

What is the value of $(a - b) \div (c + d)$,

19. When $a = {}^{+}18,\ b = {}^{-}2,\ c = {}^{+}3,\ d = {}^{+}2$?

20. When $a = {}^{-}23,\ b = {}^{+}5,\ c = {}^{-}4,\ d = {}^{-}3$?

What is the value of $a \div b \times c \div d$,

21. When $a = {}^{+}125,\ b = {}^{-}5,\ c = {}^{+}4,\ d = {}^{-}10$?

22. When $a = {}^{-}49,\ b = {}^{-}7,\ c = {}^{+}18,\ d = {}^{-}2$?

The Commutative Law for Division.

7. In a chain of divisions, or of multiplications and divisions, the successive operations are to be performed, as has been stated, in order from left to right.

E.g., $$^{-}14 \div {}^{+}2 \times {}^{-}7 = {}^{-}7 \times {}^{-}7 = {}^{+}49.$$
$$^{+}8 \div {}^{-}4 \div {}^{+}2 = {}^{-}2 \div {}^{+}2 = {}^{-}1.$$

But, if the operations in the above examples be performed in a different order, the symbol of operation, × or ÷, being carried with its proper constituent, we have

$$^{-}14 \times {}^{-}7 \div {}^{+}2 = {}^{+}98 \div {}^{+}2 = {}^{+}49, \text{ as above.}$$
$$^{+}8 \div {}^{+}2 \div {}^{-}4 = {}^{+}4 \div {}^{-}4 = {}^{-}1, \text{ as above.}$$

This example illustrates the **Commutative Law**:

(i.) *To multiply any number by a second number and then to divide the product by a third number, gives the same result as first to divide the given number by the third number and then to multiply the resulting quotient by the second number; and vice versa;* or, stated symbolically,

$$\boldsymbol{N} \times \boldsymbol{b} \div \boldsymbol{c} = \boldsymbol{N} \div \boldsymbol{c} \times \boldsymbol{b}, \text{ or } \times \boldsymbol{b} \div \boldsymbol{c} = \div \boldsymbol{c} \times \boldsymbol{b}.$$

(ii.) *If a given number be divided successively by two numbers, the result is the same whichever of the two divisions is first performed;* or, stated symbolically,

$$\boldsymbol{N} \div \boldsymbol{b} \div \boldsymbol{c} = \boldsymbol{N} \div \boldsymbol{c} \div \boldsymbol{b}, \text{ or } \div \boldsymbol{b} \div \boldsymbol{c} = \div \boldsymbol{c} \div \boldsymbol{b}.$$

These principles are proved as follows:

(i.) If in $$N \times b \div c,$$

N be replaced by $N \div c \times c$, $= N$, by (2), Art. 6, we have

$$\begin{aligned} N \times b \div c &= N \div c \times c \times b \div c \\ &= N \div c \times b \times c \div c, \text{ since } \times c \times b = \times b \times c, \\ &= N \div c \times b, \text{ since } \times c \div c = \times {}^{+}1, \text{ by (2), Art. 6.} \end{aligned}$$

In like manner, it can be shown that the Commutative Law holds for any number of successive multiplications and divisions.

The Associative Law for Division.

8. By Art. 4, ${}^{-}32 \times {}^{+}4 \div {}^{-}2 = {}^{-}128 \div {}^{-}2 = {}^{+}64$,

while, ${}^{-}32 \times ({}^{+}4 \div {}^{-}2) = {}^{-}32 \times {}^{-}2 = {}^{+}64$.

Likewise, ${}^{+}32 \div {}^{-}4 \times {}^{+}2 = {}^{-}8 \times {}^{+}2 = {}^{-}16$,

while, ${}^{+}32 \div ({}^{-}4 \div {}^{+}2) = {}^{+}32 \div {}^{-}2 = {}^{-}16$.

And, ${}^{-}32 \div {}^{-}4 \div {}^{-}2 = {}^{+}8 \div {}^{-}2 = {}^{-}4$,

while, ${}^{-}32 \div ({}^{-}4 \times {}^{-}2) = {}^{-}32 \div {}^{+}8 = {}^{-}4$.

These examples illustrate the **Associative Law**:

(i.) *A chain of multiplications and divisions may be inclosed within parentheses preceded by the symbol of multiplication, if the symbols of operation,* × *and* ÷, *preceding the numbers inclosed within the parentheses be left unchanged;* or, stated symbolically,

$$\boldsymbol{N} \times \boldsymbol{a} \div \boldsymbol{b} = \boldsymbol{N} \times (\boldsymbol{a} \div \boldsymbol{b}).$$

(ii.) *A chain of multiplications and divisions may be inclosed within parentheses preceded by the symbol of division, if the symbols of operation,* × *and* ÷, *preceding the numbers inclosed within the parentheses be reversed from* × *to* ÷ *and from* ÷ *to* ×; or, stated symbolically,

$$\boldsymbol{N} \div \boldsymbol{a} \div \boldsymbol{b} = \boldsymbol{N} \div (\boldsymbol{a} \times \boldsymbol{b}), \text{ and } \boldsymbol{N} \div \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{N} \div (\boldsymbol{a} \div \boldsymbol{b}).$$

The proof is as follows:

If in $N \div a \div b$,

N be replaced by $N \div (a \times b) \times (a \times b)$, $= N$, by (2), Art. 6,

we have
$$\begin{aligned} N \div a \div b &= N \div (a \times b) \times (a \times b) \div a \div b \\ &= N \div (a \times b) \times b \times a \div a \div b, \\ &\quad \text{since } \times a \times b = \times b \times a, \\ &= N \div (a \times b) \times b \div b, \text{ since } \times a \div a = \times {}^{+}1, \\ &= N \div (a \times b), \text{ since } \times b \div b = \times {}^{+}1. \end{aligned}$$

In like manner the other principles are proved, and all can be extended to include any number of successive multiplications and divisions.

9. An **even number** is one whose absolute value is exactly divisible by 2; as 4, 6, etc.

Since, by the Commutative Law,

$$2n \div 2 = 2 \div 2 \times n = 1 \times n = n,$$

$2n$ is always an *even* number when n is an integer.

EXERCISES X.

Find, in the most convenient way, the values of:

1. ${}^{-}25 \times {}^{-}12 \div {}^{+}5$. 2. ${}^{-}100 \times {}^{-}7 \div {}^{-}25$. 3. ${}^{-}1000 \times {}^{-}11 \div {}^{+}125$.
4. ${}^{+}33\frac{1}{3} \div {}^{-}20 \times {}^{+}3$. 5. ${}^{-}30 \div {}^{-}9 \times {}^{-}12$. 6. ${}^{-}10 \div {}^{+}17 \times {}^{-}34$.

Find, in the most convenient way, the value of $a \div b \div c \times d$,

7. When $a = {}^{+}170$, $b = {}^{-}3$, $c = {}^{+}17$, $d = {}^{-}6$.
8. When $a = {}^{-}125$, $b = {}^{-}7$, $c = {}^{+}25$, $d = {}^{-}14$.

Find the values of the following expressions, first removing the parentheses:

9. ${}^{+}25 \times ({}^{+}12 \div {}^{-}4)$. 10. ${}^{-}20 \div ({}^{-}5 \div {}^{+}2)$. 11. ${}^{+}100 \div ({}^{+}25 \times {}^{-}2)$.
12. ${}^{-}600 \div ({}^{-}200 \div {}^{-}25 \times {}^{+}3 \div {}^{-}4)$. 13. ${}^{+}300 \div ({}^{-}150 \div {}^{+}6 \times {}^{+}8 \div {}^{-}4)$.

§ 5. ONE SET OF SIGNS FOR QUALITY AND OPERATION.

1. In conformity with the usage of most text-books of Algebra we shall in subsequent work use the one set of signs, + and −, to denote both *quality* and *operation*. For the sake of brevity the sign + is usually omitted when it denotes *quality*; the sign − is never omitted.

Thus, instead of ${}^{+}2$, we shall write $+ 2$, or 2;
instead of ${}^{-}2$, we shall write -2.

2. We have used the double set of signs hitherto in order to emphasize the difference between *quality* and *operation*. It should be kept clearly in mind that the same distinction still exists.

We now have

$N + {}^{+}2 = N + (+2) = N + 2$, omitting the sign of *quality*, +;

$N + {}^{-}2 = N + (-2)$, wherein + denotes *operation*, and − denotes *quality*.

$N - {}^{+}2 = N - (+2) = N - 2$, omitting the sign of *quality*, +;

$N - {}^{-}2 = N - (-2)$, wherein the first sign, −, denotes *operation*, the second sign, −, denotes *quality*.

3. In the chain of operations

$$(+2)+(-5)-(+2)-(-11)$$

the signs within the parentheses denote *quality*, those without denote *operation*. That expression reduces to

$$(+2)-(+5)-(+2)+(+11),$$

or $2-5-2+11$, dropping the sign of *quality*, +.

In the latter expression all the signs denote *operation*, and the numbers are all *positive*.

4. The following examples illustrate the double use of the signs + and −.

Ex. **1.** ${}^{+}4+{}^{+}3=+4+(+3)=4+3=7.$

Ex. **2.** ${}^{-}5+{}^{+}2=-5+(+2)=-5+2=-3.$

Ex. **3.** ${}^{+}7-{}^{-}5=+7-(-5)=+7+(+5)=7+5=12.$

Ex. **4.** ${}^{-}4\times{}^{+}3=-4\times(+3)=-4\times 3=-12.$

Ex. **5.** ${}^{-}4\times{}^{-}3=-4\times(-3)=12.$

EXERCISES XI.

Find the values of the expressions in Exx. 1–8, first changing them into equivalent expressions in which there is only the one set of signs, + and −:

1. ${}^{+}2+{}^{+}3.$ **2.** ${}^{+}14+{}^{-}9.$ **3.** ${}^{-}8+{}^{+}5.$ **4.** ${}^{-}4+{}^{+}5.$

5. ${}^{+}8-{}^{+}4.$ **6.** ${}^{+}5-{}^{-}2.$ **7.** ${}^{-}8-{}^{+}3.$ **8.** ${}^{-}2-{}^{-}5.$

Find the values of the expressions in Exx. 9–14, first changing them into equivalent expressions in which there is only one set of signs, + and −, and then removing the parentheses:

9. ${}^{+}5+({}^{+}4+{}^{+}3).$ **10.** ${}^{+}7+({}^{-}5+{}^{+}3).$

11. ${}^{-}8-({}^{-}12+{}^{+}4).$ **12.** ${}^{+}17-({}^{+}5+{}^{-}8).$

13. ${}^{+}5-({}^{-}3+{}^{-}4)+({}^{-}2-{}^{+}8).$ **14.** ${}^{-}7-({}^{-}5-{}^{-}8)+({}^{+}6-{}^{-}7).$

Find the values of the expressions in Exx. 15–22, first changing them into equivalent expressions in which there is only one set of signs, + and −:

15. ${}^{+}3\times{}^{+}4.$ **16.** ${}^{+}18\times{}^{-}4.$ **17.** ${}^{-}9\times{}^{+}11.$ **18.** ${}^{-}4\times{}^{-}7.$

19. ${}^{+}18\div{}^{+}2.$ **20.** ${}^{+}35\div{}^{-}5.$ **21.** ${}^{-}18\div{}^{+}3.$ **22.** ${}^{-}96\div{}^{-}6.$

What are the values of $a + b - c + d - e$ and $a - (b - c + d - e)$,

23. When $a = 3,\ b = -5,\ c = -8,\ d = -9,\ e = 7$?

24. When $a = -9,\ b = 6,\ c = -9,\ d = -11,\ e = -12$?

What is the value of $a(b - c + d)$,

25. When $a = 5,\ b = -9,\ c = -11,\ d = 8$?

26. When $a = -2,\ b = 11,\ c = -12,\ d = -9$?

27. When $a = -13,\ b = -2,\ c = 3,\ d = -3$?

What is the value of $a \div (b + c - d + e)$,

28. When $a = -5,\ b = -3,\ c = 4,\ d = -5,\ e = -7$?

29. When $a = 12,\ b = -2,\ c = -9,\ d = 13,\ e = 28$?

Find the results of the following indicated operations:

30. $4 - 7$.

31. $-3 + 11$.

32. $18 - 22$.

33. $1 - 55$.

34. $2 - 4 + 6$.

35. $5 - 8 - 2$.

36. $-2 - 3 + 17 - 25 - 18 + 1$.

37. $1 - 4 + 5 - 6 + 8 - 11$.

38. -7×11.

39. $15 \times (-8)$.

40. $(-9) \times (-17)$.

41. $(-9) \times (-8)$.

42. $18 \div (-2)$.

43. $(-15) \div 5$.

44. $(-72) \div (-12)$.

45. $(-96) \times 12 \div 4$.

46. $45 \div (-5) \times 3$.

47. $(-2) \times (-3) + (-4) \times 5 - 7 \times (-3)$.

48. $5 \times (-8) - 16 \div (-4) + 2 \times (-5)$.

§ 6. POSITIVE INTEGRAL POWERS.

1. A continued product of equal factors is called a **Power** of that factor.

Thus, 2×2 is called the *second power* of 2, or 2 *raised to the second power; aaa* is called the *third power* of a, or a *raised to the third power.*

In general, $aaa \cdots$ to n factors is called the nth power of a, or a *raised to the nth power.*

The second power of a is often called the *square* of a, or a *squared;* and the third power of a the *cube* of a, or a *cubed.*

2. The notation for powers is abbreviated as follows:

a^2 is written instead of aa; a^3 instead of aaa;

a^n instead of $aaa \cdots$ to n factors.

3. The **Base** of a power is the number which is repeated as a factor.

E.g., a is the base of $a^2,\ a^3,\ \cdots,\ a^n$.

The **Exponent** of a power is the number which indicates how many times the base is used as a factor, and is written to the right and a little above the base.

E.g., the exponent of a^2 is 2, of a^3 is 3, of a^n is n.

The exponent 1 is usually omitted. Thus, $a^1 = a$.

An exponent must not be confused with a subscript. Thus, a^3 stands for the product aaa; while a_3 is a notation for a single number.

4. The definition of a *power* given above requires the exponent to be a *positive integer.* In a subsequent chapter this definition will be extended to include powers with negative and fractional exponents.

Notice that the words *positive integral* refer to the exponent and not to the value of the power, which may be negative or fractional.

E.g., $(-2)^3 = (-2)(-2)(-2) = -8$; $(\frac{2}{3})^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

5. The base of a power must be inclosed within parentheses to prevent ambiguity:

(i.) *When the base is a negative number.* Thus,

$(-5)^2 = (-5)(-5) = 25$; while $-5^2 = -(5 \times 5) = -25$.

(ii.) *When the base is a product or a quotient.* Thus,

$$(2 \times 5)^3 = (2 \times 5)(2 \times 5)(2 \times 5) = 1000;$$

while $$2 \times 5^3 = 2(5 \times 5 \times 5) = 250.$$

Likewise $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, while $\frac{2^2}{3} = \frac{2 \times 2}{3} = \frac{4}{3}$.

(iii.) *When the base is a sum.* Thus,

$$(2+3)^2 = (2+3)(2+3) = 5 \times 5 = 25;$$

while $$2 + 3^2 = 2 + 3 \times 3 = 2 + 9 = 11.$$

(iv.) *When the base is itself a power.* Thus,

$$(2^3)^2 = 2^3 \times 2^3 = (2 \times 2 \times 2)(2 \times 2 \times 2) = 64,$$

while $2^{3^2} = 2^{3 \times 3} = 2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512.$

EXERCISES XII.

Express the following powers in the abbreviated notation:

1. $2 \times 2 \times 2$. **2.** $(-3)(-3)(-3)$. **3.** $aaaaaa$.

4. $(xy)(xy)(xy)$. **5.** $(-3a)(-3a)(-3a)(-3a)$.

6. $b \cdot b \cdot b \cdots$ to 8 factors. **7.** $5 \times 5 \times 5 \cdots$ to m factors.

8. $(-ab)(-ab)(-ab) \cdots$ to n factors.

9. $3^2 \times 3^2 \times 3^2 \times 3^2 \times 3^2$. **10.** $x^n \cdot x^n \cdot x^n \cdots$ to m factors.

Express the following powers as continued products:

11. 2^6. **12.** $(-3)^7$. **13.** $(5x)^4$. **14.** $(-6z)^6$.

15. 4^m. **16.** x^n. **17.** $(ab)^p$. **18.** $(-7x)^s$.

19. $(2^2)^3$. **20.** $(-3^4)^2$. **21.** $(a^4)^2$. **22.** $[(-xy)^2]^3$.

Express the following powers in the abbreviated notation:

23. $(a+x)(a+x)(a+x)(a+x)$.

24. $(aaa-b)(aaa-b)(aaa-b) \cdots$ to 17 factors.

Express in algebraic notation:

25. The sum of the squares of a and b.

26. The square of the sum of a and b.

27. The sum of the cubes of x, y, and z.

28. The cube of the sum of x, y, and z.

Properties of Positive Integral Powers.

6. (i.) *All (even and odd) powers of positive bases are positive.*

E.g., $2^3 = 2 \times 2 \times 2 = 8$. $3^4 = 3 \times 3 \times 3 \times 3 = 81$.

In general, $(+a)^n = (+a)(+a)(+a) \cdots n$ factors.

$= +a^n$, *n even or odd.*

(ii.) *Even powers of negative bases are positive; odd powers of negative bases are negative.*

Notice that the words *even* and *odd* refer to the exponents. Also, that, for all integral values of n, $2n$ is *even* (§ 4, Art. 9), and hence that $2n+1$ or $2n-1$ is *odd*.

E.g., $(-2)^4 = (-2)(-2)(-2)(-2) = 16$;

$(-5)^3 = (-5)(-5)(-5) = -125$.

In general,

$$\begin{aligned}(-a)^{2n} &= (-a)(-a)(-a)\cdots 2\,n \text{ factors},\\ &= +(aaa\cdots 2\,n \text{ factors}), \text{ by § 3, Art. 6},\\ &= +a^{2n}.\end{aligned}$$

And,

$$\begin{aligned}(-a)^{2n+1} &= (-a)(-a)(-a)\cdots(2n+1) \text{ factors},\\ &= -[aaa\cdots(2n+1) \text{ factors}], \text{ by § 3, Art. 6},\\ &= -a^{2n+1}.\end{aligned}$$

7. If $a = b$, then $a^n = b^n$. This principle follows directly from the axioms.

EXERCISES XIII.

Find the values of the following powers:

1. 2^3. **2.** 3^2. **3.** $(-3)^5$. **4.** -3^5. **5.** $(-2)^6$.

6. -2^6. **7.** $(-5)^4$. **8.** -5^4. **9.** $(-x)^4$. **10.** $(-x)^7$.

Express as powers of 2:

11. 8. **12.** 64. **13.** 512. **14.** 4096.

Express as powers of -3:

15. 9. **16.** 81. **17.** -243. **18.** 729.

Express as products of powers of 2 and 3:

19. 24. **20.** 144. **21.** 1536. **22.** 2916.

Determine, by inspection, the signs of the following powers:

23. $(-5)^{17}$. **24.** $(-7)^{76}$. **25.** $(-3)^{2p}$. **26.** $(-4)^{2p+1}$.

27. $(-7)^{n-1}$, when n is even. **28.** $(-3)^n$, when n is even.

Find the values of the following expressions:

29. $2^2 + 3^2$. **30.** $(2+3)^2$. **31.** $(3^3 - 5^3)$. **32.** $(3-5)^3$.

33. 5×4^3. **34.** $(5 \times 4)^3$. **35.** $2(-7)^3$. **36.** $[2(-7)]^3$.

37. $2 \times 3^4 - (3 \times 2)^4$. **38.** $(5 \times 6)^2 - 6 \times 5^2$. **39.** $2^7 - (-2)^7$.

40. $29^{24} - (-29)^{24}$. **41.** $[9(-4)]^{11} + 36^{11}$. **42.** $(-37)^{13} + 37^{13}$.

When $a = 1$, $b = -3$, $c = 2$, find the values of:

43. a^c. **44.** b^a. **45.** ab^c. **46.** $(ab)^c$. **47.** $(b^a)^c$.

48. b^{a^c}. **49.** $a^2 + b^2 + c^2$. **50.** $(ab)^3 + (bc)^3 + (ac)^3$.

Find the values of the following sums when $n = 5$, $a = 2$, $b = -3$:

51. $1^2 + 2^2 + 3^2 + \cdots + n^2$. **52.** $(1 + 2 + 3 + \cdots + n)^3$.

53. $a + a^2 + a^3 + \cdots + a^n$. **54.** $b + b^2 + b^3 + \cdots + b^n$.

CHAPTER III.

THE FUNDAMENTAL OPERATIONS WITH INTEGRAL ALGEBRAIC EXPRESSIONS.

§ 1. DEFINITIONS.

1. An **Integral Algebraic Expression** is an expression which involves only additions, subtractions, multiplications, and positive integral powers of *literal* numbers; that is, in which the *literal* numbers are connected only by one or more of the symbols of operation, +, −, ×, but not by the symbol ÷.

E.g., $1 + x + x^2$, $5\,a^2b + \frac{2}{3}\,cd^2$, etc.

The word *integral* refers only to the *literal* parts of the expression. At the same time, the letters are not *limited to integral numerical values, but, as always, may have any values whatever.*

E.g., $a + b$ is *algebraically* integral; but when $a = \frac{1}{2}$, $b = \frac{3}{4}$, we have

$$a + b = \tfrac{1}{2} + \tfrac{3}{4} = 1\tfrac{1}{4}.$$

2. Coefficients. — In a product, any factor, or product of factors, is called the **Coefficient** of the product of the remaining factors.

E.g., in $3\,abc$, 3 is the coefficient of abc, $3\,b$ of ac, etc.

A **Numerical Coefficient** is a coefficient expressed in figures.

E.g., in $-3\,ab$, -3 is the numerical coefficient of ab.

A **Literal Coefficient** is a coefficient expressed in letters, or in letters and figures.

E.g., in $3\,ab$, a is the literal coefficient of $3\,b$, and $3\,a$ of b.

The coefficients $+1$ and -1 are usually omitted.

3. The sign + or the sign −, preceding a product, is to be regarded as the sign of its numerical coefficient.

Thus, $+3a$ means the product of *positive* 3 by a; $-5x$ means the product of *negative* 5 by x. In particular, $+a$ means the product of *positive* 1 by a, and $-a$ means the product of *negative* 1 by a, unless the contrary is stated.

EXERCISES I.

What is the coefficient of x in

1. $2x$? **2.** $-3x$? **3.** $5ax$? **4.** $-7bx$? **5.** $(a+b)x$?

6. If the sum, $a+a+a+a$, be represented as a product, what is the coefficient of a?

7. If the algebraic sum, $-b-b-b-b-b$, be represented as a product, what is the coefficient of $-b$? Of b?

8. If the sum, $2ax+2ax+2ax+\cdots$ to y summands, be represented as a product, what is the coefficient of $2ax$? Of $2a$? Of ay?

4. Terms. — In the expression $4a-3b$, the sign $-$ means *operation*, i.e., *subtraction*.

But, since $$4a-3b=4a+(-3b),$$
the given expression is the result of adding $4a$ and $-3b$.

Upon these considerations are based the following definitions:

The **Terms** of an algebraic expression are the *additive* and *subtractive* parts of that expression.

E.g., the terms of $4a-3b$ are $4a$ and $-3b$.

The **Sign of a Term** is the sign of quality, $+$ or $-$, of its numerical coefficient.

E.g., the sign of the term $4a$ is $+$, of $-3b$ is $-$.

A **Positive Term** is one whose sign is $+$; as $4a$.

A **Negative Term** is one whose sign is $-$; as $-3b$.

5. Like or Similar Terms are terms which do not differ, or which differ only in their numerical coefficients.

E.g., in the expression $+3a+6ab-5a+7ab$, $+3a$ and $-5a$ are like terms; so are $+6ab$ and $+7ab$.

Unlike or Dissimilar Terms are terms which are not like.

E.g., $+3a$ and $+7ab$ in the above expression.

6. A **Monomial** is an expression of one term; as a, $-7\,bc$.

A **Binomial** is an expression of two terms; as $-2\,a^2+3\,bc$.

A **Trinomial** is an expression of three terms.

E.g., $a+b-c$, $-3\,a^2+7\,b^3-5\,c^4$.

A **Multinomial*** is an expression of two or more terms, including, therefore, binomials and trinomials as particular cases.

E.g., $a+b^2$, a^2+b-c^3, $ab+bc-cd-ef$.

§ 2. ADDITION AND SUBTRACTION.

1. Like Terms can be united by addition and subtraction into a single *like* term.

Just as $2=1+1$, so $2\,xy=xy+xy$;
just as $-3=-1-1-1$, so $-3\,xy=-xy-xy-xy$.

Therefore, just as $2-(-3)=2+3=5$,

so $$2\,xy-(-3\,xy)=[2-(-3)]\,xy=5\,xy.$$

That is, *to add or subtract like terms, add or subtract their numerical coefficients and annex to that result their common literal part.*

Ex. 1. Add $-7\,ab$ to $4\,ab$.

We have $4\,ab+(-7\,ab)=[4+(-7)]\,ab=-3\,ab$.

Ex. 2. Find the sum of $3\,a$, $-5\,a$, $8\,a$, $-4\,a$.

By the Commutative Law for addition

$$3\,a+8\,a+(-5\,a)+(-4\,a)=[3+8+(-5)+(-4)]\,a=2\,a.$$

Ex. 3. Subtract $-5\,x^2y$ from $-7\,x^2y$.

We have

$$-7\,x^2y-(-5\,x^2y)=-7\,x^2y+5\,x^2y=(-7+5)\,x^2y=-2\,x^2y.$$

EXERCISES II.

Add

1. $2\,a$ to $3\,a$. 2. $-7\,y$ to $3\,y$. 3. $4\,b$ to $-9\,b$.
4. $-6\,c$ to $-5\,c$. 5. $-4\,x^3$ to $\frac{1}{3}\,x^3$. 6. $\frac{15}{2}\,a^2$ to $\frac{7}{3}\,a^2$.

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* The word **Polynomial** is frequently used instead of **Multinomial**.

Subtract

7. $2a$ from $5a$. 8. $3b$ from $-5b$. 9. $10m$ from $-m$.
10. $-9y$ from $2y$. 11. $-16x^2$ from $-5x^2$. 12. $-3ab$ from $7ab$.
13. $-7x^{n-1}(y-z)$ from $2x^{n-1}(y-z)$.

Find the sum of

14. $a, 2a, -3a$. 15. $-ab, -3ab, -7ab$.
16. $3x^n, -4x^n, -9x^n$. 17. $2a^2b^2, -a^2b^2, -5a^2b^2$.
18. $-5ax^m, 7ax^m, -9ax^m, 3ax^m$.
19. $(x^n+y^n), -3(x^n+y^n), 4(x^n+y^n), -7(x^n+y^n), 15(x^n+y^n)$.

Simplify the following expressions:

20. $5x-2x+4x$. 21. $-9b-2b-3b$. 22. $-7m+4m-5m$.
23. $-x^{2n}+7x^{2n}+2x^{2n}-5x^{2n}$. 24. $a^2b-2ba^2-3a^2b+4ba^2$.
25. $-(a+b-c)+2(a+b-c)+11(a+b-c)-7(a+b-c)$.
26. $3x+(-5x)$. 27. $-10a+(-12a)$. 28. $12x^2-(-7x^2)$.
29. $2a-[-4a-(-6a)]$. 30. $m+[2m-(3m-4m)]$.
31. $6y-[5y-4y-(-3y+2y)]-y$.
32. $x-[x-2x-(x-3x)-(x-4x)]$.

2. **Unlike Terms** are added and subtracted by writing them in succession, each preceded by the sign + if it is to be added, by the sign − if it is to be subtracted.

Ex. 1. Add $3b$ to $2a$. We have $2a+3b$.

Ex. 2. Add $-3x^2$ to $2y^2$. We have

$$2y^2+(-3x^2)=2y^2-3x^2.$$

Ex. 3. Subtract $-11m$ from $2n$. We have

$$2n-(-11m)=2n+11m.$$

Such steps as changing $+(-3x^2)$ into $-3x^2$, and $-(-11m)$ into $+11m$ should be performed mentally.

3. A multinomial consisting of two or more sets of like terms can be simplified by uniting like terms.

Ex. 1. $2a-3b-5a+4b$

$=2a-5a-3b+4b$, by the Commutative Law,

$=-3a+b$, by the Associative Law.

EXERCISES III.

Add

1. a to 1. **2.** -3 to $2x$. **3.** $-x$ to $-y$.

4. x^2 to $-y^2$. **5.** a^2 to a. **6.** $-7z$ to $-3x$.

7. $-2ab$ to a^2. **8.** $-xy^2$ to $-x^2y$. **9.** $-3m^n$ to $2n^m$.

Subtract

10. x from 1. **11.** $-a$ from 2. **12.** $-2m$ from $-3n$

13. $-x^3$ from $3x$. **14.** $-bx$ from $-cy$. **15.** $7x^m$ from $-2y^n$.

Add

16. $1, -x, x^2$. **17.** $-3, 2x, -3y$. **18.** $-ab, -ac, -ad$.

19. Subtract $-3x^2$ from the sum of 2 and $-4x$.

20. Add $-5x^2$ to the result of subtracting $-2x$ from 0.

Simplify the following expressions by uniting like terms:

21. $a+1+a-1$. **22.** $2x+5+3x-7$.

23. $-5mn+3n-2nm-6n$. **24.** $3x^2-4y^2+2x^2-6y^2$.

25. $a+b-3a+c-4b+6a-5c-8a-3c+11b$.

26. $6m^4-cm^4+3+9cm^4-8m^4+5m^4-m^4c+11$.

27. $9ab^4-bx-13ab^4-a^4b+3bx-2ab^4+10a^4b-2bx-ab^4$.

28. $3(a+m)-4(a+m)-2(a+m)+8(a+m)$.

29. $(a+z)^3-2(a+z)^3+2(a+z^3)+7(a+z)^3-5(a+z^3)$.

Simplify the following expressions, first removing parentheses:

30. $a+1-(2-3a)$. **31.** $5x-(-2y+3x)$.

32. $2m+3n-(5m-4n)-(-3m+7n)$.

33. $1-[a^3-2-(-2a^3-3)]$.

34. $x^2-y^2+[-3x^2-2y^2-(2x^2-3y^2)]$.

35. $2xy+5yz-(2xy-3yz)-[2xy-(3xy-2yz)+5yz]$.

Find the values of the expressions in Exx. 30–35,

36. When $a=1$, $x=3$, $y=-5$, $z=10$, $m=4$, $n=-7$.

37. When $a=-3$, $x=6$, $y=-7$, $z=8$, $m=-1$, $n=-2$.

Simplify

38. $a+(a+1)+(a+2)+(a+3)$.

39. $x+(x+2)+(x+4)+\cdots+(x+10)$.

40. $2m+(2m-1)+(2m-2)+\cdots+(2m-9)$.

41. Find the sum of 7 terms, the first term being x^2, and each succeeding term being 1 less than the preceding term.

42. Find the sum of 6 terms, the first term being $m+n$, and each succeeding term being p less than the preceding term.

Addition and Subtraction of Multinomials.

4. Ex. 1. Add $-2a+3b$ to $3a-5b$.

We have
$$\begin{aligned}(3a-5b)+(-2a+3b)&=3a-5b-2a+3b,\\&=a-2b.\end{aligned}$$

Ex. 2. Subtract $-2a+3b$ from $3a-5b$.

We have
$$\begin{aligned}(3a-5b)-(-2a+3b)&=3a-5b+2a-3b,\\&=5a-8b.\end{aligned}$$

In adding multinomials, it is often convenient to write one underneath the other, placing like terms in the same column.

Ex. 3. Find the sum of $-4x^2+3y^2-8z^2$, $2x^2-3z^2$, and $2y^2+5z^2$.

We have
$$\begin{array}{rrr}-4x^2&+3y^2&-8z^2\\2x^2&&-3z^2\\&2y^2&+5z^2\\\hline-2x^2&+5y^2&-6z^2\end{array}$$

It is evidently immaterial whether the addition is performed from left to right, or from right to left, since there is no carrying as in arithmetical addition.

Ex. 4. Subtract $-2a+3b$ from $3a-5b$.

Changing mentally the signs of the terms of the subtrahend, and adding, we have
$$\begin{array}{rr}3a&-5b\\-2a&+3b\\\hline5a&-8b\end{array}$$

Ex. 5. Subtract $2x^2-3z^2$ from $-4x^2+3y^2$, and from the result subtract $2y^2+5z^2$.

When several multinomials are to be subtracted in succession, the work is simplified by writing them with the signs of the terms already changed. We then have
$$\begin{array}{rrr}-4x^2&+3y^2&\\-2x^2&&+3z^2\\&-2y^2&-5z^2\\\hline-6x^2&+\ y^2&-2z^2\end{array}$$

EXERCISES IV.

Add

1. $a + 4$ to $a - 4$.
2. $-x + y$ to $x - y$.
3. $7a - 4b$ to $-3a + 2b$.
4. $2x^2 - xy$ to $-x^2 + y^2$.
5. $x^3 + x + 1$ to $x^2 - x + 1$.
6. $2a^2 - 3ab - b^2$ to $-a^2 + 5ab + 2b^2$.

Subtract

7. $a - 1$ from $a + 1$.
8. $a - 2b$ from 0.
9. $8a - 3b$ from $7a - 2b$.
10. $x - y$ from $x + y$.
11. $-x^2y - xy^2$ from $x^3 + y^3$.
12. $a^3b + ab^3$ from $a^4 + b^4$.
13. $a^3 - 2a^2 - a - 2$ from $a - 5$.
14. $-x^4 + 7x^3 + 3x^2 + 3x - 9$ from 0.
15. $x^3 - 3x^2y + 3xy^2 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$.
16. $2x^4 - 3x^3 - 7x^2 + 3x + 1$ from $2 + 4x - 6x^2 - 2x^3 + 3x^4$.
17. $x^4 - x^3 + x^2 - x + 1$ from $x^4 + x^2 + 1$.
18. $-\frac{1}{2}x^5 - \frac{2}{3}x^3 + \frac{3}{5}x^2 - 1$ from $x^5 + \frac{1}{2}x^4 - x^3 - \frac{2}{3}$.
19. $-3(x + y)^2 + 4(x + y) - 7$ from $-(x + y)^2 - 7(x + y) + 3$.

Find the sum of

20. $7a - 9b - c,\ 5a - 3b - 2c,\ 2a + 3b - 5c$.
21. $3x^2 - 5x + 1,\ 7x^2 + 2x - 3,\ -x^2 - 2x - 3$.
22. $x^2 - ax + a^2,\ 2x^2 + 3ax - 4a^2,\ x^2 + ax + 2a^2$.
23. $3a^2 - 4ab + b^2,\ a^2 - 2ab - 2b^2,\ 2a^2 - 3ab + 4b^2$.
24. $a^2 - 2ab + 2b^2,\ 2a^3 - 3ab + b^3,\ a^2 + 5ab - b^3$.
25. $2x^2y^3 + 4x^3y^2,\ -5x^2y^3 + 2x^2y^2 - 3x^3y^2,\ 4x^3y^2 - 5x^2y^3 - 6x^2y^2$.
26. $3a - 2b + 5c,\ a + b - c,\ -2a + 5b - 3c,\ -2a + b - c$.
27. $x^3 + 2x^2 - 3x + 1,\ 2x^3 - 3x^2 + 4x - 2,\ 5x^3 + 4x^2 + 5,$
    $6x^3 - 5x^2 - 4x - 3$.
28. $5a^3 - 3a^2 + 2a,\ a^3 - a^2,\ a^2 - a + 1,\ a^3 - 2a^2 - a - 2$.
29. $2a + b - (c + d),\ a + (b - c) - d,\ a + b - (c - d)$.
30. $a + 2b + c,\ 2a - (b - c) - d,\ 3a + b - (2c + d)$.
31. $3(a + b) - 4(a + b)^2 + 5(a + b)^3,\ (a + b)^2 - 2(a + b)^3,$
    $-(a + b)^3 + 2(a + b)^2 - (a + b)$.
32. $7(x^2 + y^2) - 3(x^2 - y^2) + 2xy,\ 2(x^2 - y^2) - 4xy,\ 3(x^2 + y^2) - (x^2 - y^2)$.
33. Subtract the sum of $a^2 + ab + b^2$ and ab from $2a^2 + 3ab + 2b^2$.
34. Subtract the sum of a^2 and $b^2 + c^2$ from the sum of b^2 and $a^2 - c^2$

35. How much does $m^2 + n^2$ exceed $m^2 - n^2$?

36. How much does $1 - x^2$ exceed $2 - 3x^2$?

37. What expression must be added to $2a - 3b + 4c$ to give $4a + 2b - 2c$?

38. What expression must be added to $xy + xz + yz$ to give $x^2 + y^2 + z^2$?

39. What expression must be subtracted from $a^2 + ab + b^2$ to give $a^2 - 2ab + b^2$?

40. What expression must be subtracted from $x^2 - 2xy + y^2$ to give $x^2 + 2xy + y^2$?

41. What expression must be added to $x^2 + x + 1$ to give 0?

If $x = 2a - 3b + 4c$, $y = -3a + 2b - 7c$, $z = 9a - 7b + 6c$, find the values of

42. $x + y + z$. **43.** $x - y + z$. **44.** $x + y - z$. **45.** $x - y - z$.

Given the four expressions:

$$\begin{aligned} x &= 5a^2 - 3ab + b^2 - 3ac + 2bc + c^2, \\ y &= 2a^2 + 5ab - 3b^2 + 2ac - 4bc + 3c^2, \\ z &= 4a^2 - 7ab + 5b^2 - 4ac - 5bc + c^2, \\ u &= 2a^2 + 9ab - 8b^2 + 3ac + 3bc + 2c^2, \end{aligned}$$

find the values of

46. $x + y - z + u$. **47.** $x + y - z - u$. **48.** $x - y - z - u$.

§ 3. MULTIPLICATION.

Principles of Powers.

1. Products of Powers.

Ex. $xx^2x^3 = (x)(xx)(xxx) = xxxxxx = x^6 = x^{1+2+3}$.

This example illustrates the following principle:

The product of two or more powers of one and the same base is equal to a power of that base whose exponent is the sum of the exponents of the given powers; or, stated symbolically,

$$a^m a^n = a^{m+n};\quad a^m a^n a^p = a^{m+n+p};\ \text{etc.}$$

For,

$$\begin{aligned} a^m a^n &= (aaa \cdots \text{to } m \text{ factors})(aaa \cdots \text{to } n \text{ factors}) \\ &= aaa \cdots \text{to } (m+n) \text{ factors} = a^{m+n}. \\ a^m a^n a^p &= (a^m a^n)a^p = a^{m+n}a^p = a^{m+n+p}. \end{aligned}$$

EXERCISES V.

Express each of the following products as a single power:

1. 2×2^2. **2.** $(-7)^3(-7)^5$. **3.** $(-2)^3 2^4$.

4. $5^4(-5)^6$. **5.** a^2a^3. **6.** $(-x)^3x^4$.

7. $(ab)^2(ab)^3$. **8.** $a^2a^3a^5$. **9.** $(-x)(-x)^2(-x)^3$.

10. $a^6a^pa^q$. **11.** $a^{m-1}a^{m+2}$. **12.** $x^{m+n}x^{m-3n}$.

13. $(a^2+b^2)^3(a^2+b^2)^5$. **14.** $(x+y)^n(x+y)^2$.

2. Powers of Powers.

Ex. $(a^4)^5 = a^4a^4a^4a^4a^4 = a^{4+4+4+4+4} = a^{4\times 5} = a^{20}$.

This example illustrates the following principle:

A power of a power of a given base is equal to a power of that base whose exponent is the product of the given exponents; or, stated symbolically,

$$(a^m)^n = a^{mn};\ [(a^m)^n]^p = a^{mnp};\ \text{etc.}$$

For, $(a^m)^n = a^ma^ma^m \cdots \text{to } n \text{ factors}$

$= a^{m+m+m+\cdots \text{ to } n \text{ summands}} = a^{mn}$.

Likewise, $[(a^m)^n]^p = (a^{mn})^p = a^{mnp}$; and so on.

EXERCISES VI.

Find the values of the following powers:

1. $(3^2)^3$. **2.** 3^{2^3}. **3.** $(4^3)^2$. **4.** $[(-2)^3]^4$. **5.** $(-2^3)^5$.

Simplify the following powers:

6. $(11^4)^5$. **7.** $[(-18)^5]^6$. **8.** $[(2^3)^2]^4$. **9.** $(a^3)^4$.

10. $(-x^4)^3$. **11.** $[(-x)^4]^3$. **12.** $[(ab)^2]^5$. **13.** $[(x^2)^5]^7$.

14. $[(-n^2)^5]^3$. **15.** $(x^n)^{2n}$. **16.** $(x^3)^{2n}$. **17.** $[(x+y)^3]^2$.

Express the following powers as powers of 2:

18. $[(2^3)^2]^4$. **19.** $(2^{3^2})^4$. **20.** 8^5. **21.** 32^{25}.

Express the following powers as powers of 3^2:

22. 3^8. **23.** 3^{2a}. **24.** 9^4. **25.** 27^6.

Express the following powers as powers of 5^3:

26. 5^{15}. **27.** 5^{81}. **28.** 125^6. **29.** 25^9.

Simplify the following powers:

30. $(a^2a^3)^4$. **31.** $(x^2x^5)^6$. **32.** $[(-x)^2x^7]^3$. **33.** $[(-c)^3c^4]^{5n}$.

Write the squares and the cubes of:

34. a^2. **35.** $-a^3$. **36.** $(x^2x^3)^3$. **37.** $[(-y)^3y^4]^5$.

38. $x+y$. **39.** $(a-b)^2$. **40.** $-(a+b-c)^3$.

Write

41. The fourth power of a. **42.** The ath power of 4.

Write the sum of *ten* terms, the first term being x,

43. When each term is the square of the preceding term.

44. When each term is the nth power of the preceding term.

Simplify

45. $3(a^3)^4+2(a^4)^3-4(a^2)^6$. **46.** $3(a^5)^4-2(a^4)^5-5(a^{10})^2+7[(a^2)^5]^2$.

3. Like and Unlike Powers. — Two powers are said to be *like* or *unlike* according as their exponents are equal or unequal, whether or not their bases are equal. Thus,

a^2, b^2 are *like* powers; a^2, a^3, a^4 are *unlike* powers.

4. Products of Like Powers.

Ex. $a^4b^4c^4 = (aaaa)(bbbb)(cccc)$

$= (abc)(abc)(abc)(abc)$, by the Commutative Law,

$= (abc)^4$, by the definition of a power.

This example illustrates the following principle:

(i.) *The product of like powers of two or more given bases is the like power of the product of the bases;* or, stated symbolically,

$$a^mb^m = (ab)^m;\quad a^mb^mc^m = (abc)^m, \text{ etc.}$$

For, $a^mb^m = (aaa \cdots \text{to } m \text{ factors})(bbb \cdots \text{to } m \text{ factors})$

$= (ab)(ab)(ab) \cdots$ to m factors, by the Commutative Law,

$= (ab)^m$, by the definition of a power.

In like manner the principle can be extended to the product of any number of like powers.

(ii.) *The converse of the principle is evidently true:*

$$(ab)^m = a^mb^m;\quad (abc)^m = a^mb^mc^m, \text{ etc.}$$

E.g., $(xy)^3 = x^3y^3$; $(2a^2b)^3 = 2^3(a^2)^3b^3 = 8a^6b^3$.

EXERCISES VII.

Express the following products of powers as powers of products:

1. $7^3 \times 5^3$. **2.** $(8)^4 \times (-3)^4$. **3.** a^3b^3. **4.** $(-x)^3y^3$.

5. $(-a)^5b^5(-c)^5$. **6.** $a^2(b+c)^2$. **7.** $a^6b^3c^9$. **8.** $x^{12}y^{15}z^{18}$.

Express the following powers of products as products of powers, reducing powers of any numerical factors:

9. $(xy)^3$. **10.** $(-2u)^5$. **11.** $(-2xy)^4$. **12.** $(abc)^3$.

13. $(a^2b^3c)^4$. **14.** $(-3x^2y)^4$. **15.** $(-x^2y^3z)^5$. **16.** $(m^2n^3)^p$.

Write

17. The square of twice a. **18.** Twice the square of a.

19. Four times the square of the difference between x and y.

20. The square of four times the difference between x and y.

Given two numbers, a and b, write in algebraic language:

21. The square of the first number, plus twice the product of the two numbers, plus the square of the second number.

22. The cube of the first number, plus three times the product of the square of the first by the second, plus three times the product of the first by the square of the second, plus the cube of the second.

23. Write in algebraic language the verbal statements in Exx. 21 and 22, when the given numbers are $2a$ and $-3b$.

Degree. Homogeneous Expressions.

5. An integral term which is the product of n letters is said to be of the *nth degree*, or of *n dimensions*.

Thus, the **Degree of an Integral Term** is indicated by the sum of the exponents of its literal factors.

E.g., $3ab$ is of the *second* degree; $2x^2y, = 2xxy$, is of the third degree.

The **Degree of a Multinomial** is the degree of that term which is of highest degree.

E.g., the degree of $x^2y + xy^3 - x^2y^3z$ is the degree of x^2y^3z; *i.e.*, the *sixth*.

6. It is often desirable to speak of the degree of a term, or of an expression, in regard to one or more of its literal factors.

E.g., the term ax^2y^3 is of the *fifth* degree in x and y, of the *first* degree in a, of the *second* degree in x, of the *third* degree in y, etc.

The expression $ax^2 + 2\,bxy + cy^2$ is of the *second* degree in x, in y, and in x and y.

7. A **Homogeneous Expression** in one or more letters is an expression all of whose terms are of the same degree in these letters.

E.g., $a^2 + 2\,ab + b^2$ is homogeneous in a and b.

8. If the terms of a multinomial be arranged so that the exponents of some one letter increase, or decrease, from term to term, the multinomial is said to be arranged to *ascending*, or *descending*, powers of that letter. The letter is called the *letter of arrangement*.

E.g., $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ is arranged to *descending* powers of a, which is then the letter of arrangement; or to *ascending* powers of b, which is then the letter of arrangement.

EXERCISES VIII.

What is the degree of $2\,a^3b^2x^4y^5$

1. In a? **2.** In x? **3.** In b and y? **4.** In a, b, x, and y?

What is the degree of the expression $a^3x^4 - 6\,a^2b^2x^3y + 5\,abx^2y^2$

5. In x? **6.** In y? **7.** In a? **8.** In b?

9. Arrange $2\,x - 3\,x^5 + 7 - 2\,x^4 + 3\,x^2$ to ascending powers of x; to descending powers of x.

10. Arrange $3\,y - 7\,xy^3 + 5\,x^3y^2 + 4\,x^2y^4$ to ascending powers of x; to ascending powers of y.

11. Arrange $29\,a^2b^4 + 4\,b^6 - 30\,a^3b^3 + 25\,a^4b^2 - 12\,ab^5$ to descending powers of a; to descending powers of b.

Multiplication of Monomials by Monomials.

9. Ex. 1. $3\,a \times 5\,b = 3 \times 5 \times a \times b = 15\,ab.$

Ex. 2. $2\,x \times (-4\,y^2) = 2(-4)\,xy^2 = -8\,xy^2.$

Ex. 3. $\frac{2}{3}\,a^3 \times 6\,ab^2 \times 11\,b^5 = \frac{2}{3} \times 6 \times 11 \times a^3ab^2b^5 = 44\,a^4b^7.$

Ex. 4. $3\,a^{m+1}b^2 \times 5\,a^3b^{n-1} = 3 \times 5\,a^{m+1}a^3b^2b^{n-1} = 15\,a^{m+4}b^{1+n}.$

These examples illustrate the following method:

The product of two or more monomials is obtained by multiplying the product of their numerical coefficients by the product of their literal factors.

EXERCISES IX.

Multiply

1. $3a$ by 4. **2.** -5 by $-2a$. **3.** 7 by $-5x^2$.

4. $2a$ by $3a$. **5.** $2\frac{1}{2}x$ by $-5x^3$. **6.** $-3a^2$ by $-4a$.

7. $-2ab$ by $5ab$. **8.** $3a^2b$ by $-7ab^2$. **9.** $4b^2c$ by $-3b^3c^2$.

10. $3abc$ by ab^2c^3. **11.** $-3x^2yz$ by xy^2. **12.** a^2bxy by $ab^2x^2y^2$.

13. $2(a+b)^3$ by $-3(a+b)^2$. **14.** $7\frac{1}{2}a^2(x-y)^3$ by $6a^3(x-y)^2$.

15. $12a^4b^m$ by $-\frac{3}{4}ab^n$. **16.** $3a^{m-1}b^{n+1}$ by $-12a^5b^3$.

Simplify the following continued products:

17. $3ab \times 5bc \times 6ac$. **18.** $-7x^2y \times (-2y^2z) \times 3xz^2$.

19. $-axy \times 7abx^2z \times 2bx^2yz$. **20.** $x^2y^{n+1} \times 5x^my^{2n} \times (-x^{5m}y^{2n-1})$.

21. $(2ax^3)^4 \times (5ab^2xy)^2 \times (-2a^2x^2y^3)^3$.

22. $(1-x)^3 \times 3ab \times 4(1-x)^5 \times (-2a^2c)$.

The Distributive Law for Multiplication.

10. If the indicated operation within the parentheses in the product, $4(2+3)$, be first performed, we have

$$4(2+3) = 4 \times 5 = 20.$$

But if each term within the parentheses be multiplied by 4 and the resulting products be then added, we have

$$4 \times 2 + 4 \times 3 = 8 + 12 = 20, \text{ as above.}$$

Therefore $\quad 4(2+3) = 4 \times 2 + 4 \times 3.$

This example illustrates the following principle:

The Distributive Law. — *The product of a multinomial by a monomial is obtained by multiplying each term of the multinomial by the monomial and adding algebraically the resulting products.* That is,

$$a(b+c-d) = ab + ac - ad.$$

(i.) For, let a be limited to positive integral values.

Then

$$\begin{aligned} a(b+c-d) &= (b+c-d)+(b+c-d)+\cdots \text{ to } a \text{ summands,}\\ &= (b+b+\cdots \text{ to } a \text{ summands})+(c+c+\cdots \text{ to } a \text{ summands})\\ &\qquad -(d+d+\cdots \text{ to } a \text{ summands}),\\ &= ab+ac-ad. \end{aligned}$$

(ii.) Let a be limited to negative integral values, and be denoted by $-x$, so that x is an absolute number.

Then

$$\begin{aligned} a(b+c-d) &= -x(b+c-d), \text{ replacing } a \text{ by } -x,\\ &= -(b+c-d)-(b+c-d)-\cdots \text{ to } x \text{ summands,}\\ &= -b-b-\cdots \text{ to } x \text{ summands } -c-c-\cdots \text{ to } x \text{ summands}\\ &\quad +d+d+\cdots \text{ to } x \text{ summands,}\\ &= +(-b-b-\cdots \text{ to } x \text{ summands})+(-c-c-\cdots \text{ to } x \text{ summands})\\ &\quad -(-d-d-\cdots \text{ to } x \text{ summands}),\\ &= +(-x)b+(-x)c-(-x)d\\ &= ab+ac-ad, \text{ replacing } -x \text{ by } a. \end{aligned}$$

In (i.) and (ii.) a was limited to integral numerical values. Similar reasoning can, however, be applied when a is a numerical fraction.

Thus, $\frac{2}{3}(4+\frac{5}{7})=\frac{4+\frac{5}{7}}{3}+\frac{4+\frac{5}{7}}{3}=\frac{4}{3}+\frac{4}{3}+\frac{\frac{5}{7}}{3}+\frac{\frac{5}{7}}{3}=\frac{2}{3}\times 4+\frac{2}{3}\times\frac{5}{7}.$

Multiplication of a Multinomial by a Monomial.

11. The multiplication of a multinomial by a monomial is a direct application of the Distributive Law.

Ex. 1. Multiply $(x-y)$ by 3.

We have $3(x-y)=3x-3y.$

Ex. 2. Multiply $3x-2y-7z$ by $-4x$.

We have

$$\begin{aligned} -4x(3x-2y-7z) &= (-4x)(3x)-(-4x)(2y)-(-4x)(7z)\\ &= -12x^2+8xy+28xz. \end{aligned}$$

Such steps as changing $(-4x)(3x)$ into $-12x^2$, $-(-4x)(2y)$ into $+8xy$, and $-(-4x)(7z)$ into $+28xz$, should be performed mentally.

Ex. 3. $4x^2y(xy-3xz+2yz)=4x^3y^2-12x^3yz+8x^2y^2z.$

EXERCISES X.

Multiply

1. $a + 1$ by 3. **2.** $2a - 5$ by -4. **3.** $2a - 3b$ by $-2a$.
4. $7x - 8y$ by $3x$. **5.** $5a^2 - 3ab$ by $2a^2b$.

Simplify the following expressions:

6. $2a - 3(a - 1)$. **7.** $3x - 2(3x - 2)$. **8.** $a^2 - a(a - 1)$.
9. $5a + 2a(a - 1) - 3a(a + 1)$. **10.** $1 - [5(a - b) + 6(a + b)]$.
11. $5x - 3(x - 2y) - 7[5x - 3(x - 3y)]$.
12. $a + a(1 + a^2) - a[1 - a(1 - a)]$.

Multiply $a^3b - 3ab^2c + 4bc^2$ by

13. $2ab$. **14.** $-3ac$. **15.** a^2bc^{m-1}. **16.** $-\frac{2}{3}ab^2c^3$.

Multiply $x^3 - 2x^2 + 6x - 1$ by

17. -2. **18.** $3x$. **19.** $-5x^2$. **20.** $\frac{2}{3}x^3$.

21. Multiply $(x^2 + 1)^4 - 5a(x^2 + 1)^2 + 3ab$ by $-2a^2b^2(x^2 + 1)^3$.

Simplify the result of substituting $a + b - c$ for x, and $a - b + c$ for y, in the following expressions:

22. $5bx - 7ay$. **23.** $3a^2bx - 14ab^2y$. **24.** $7abx + 2bcy$.

Find the values of the results of Exx. 22–24,

25. When $a = -2$, $b = 3$, $c = -4$.
26. When $a = 5$; $b = -7$, $c = -5\frac{1}{2}$.

Multiply $5x^n - 3x^{n-3}y^2 + 4x^{n-5}y^4 + y^{n-4}$ by

27. x^3. **28.** $-5x^2y$. **29.** $3x^ny^4$. **30.** $-6\frac{2}{3}x^ny^m$.

Write the squares, the cubes, and the nth powers of:

31. a^{m+1}. **32.** x^{m-2}. **33.** $2x^{m+n}y$. **34.** $-3a^{m+n-1}y^3$.

The Distributive Law when the Multiplier is a Multinomial.

12. Ex. $(2 + 3)(7 - 5) = (2 + 3)7 - (2 + 3)5$

$$= 2 \times 7 + 3 \times 7 - 2 \times 5 - 3 \times 5.$$

In general

$$(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{c} + \boldsymbol{d} - \boldsymbol{e}) = \boldsymbol{ac} + \boldsymbol{bc} + \boldsymbol{ad} + \boldsymbol{bd} - \boldsymbol{ae} - \boldsymbol{be}, \text{ etc.}$$

For, $(a + b)(c + d - e) = (a + b)c + (a + b)d - (a + b)e$, by Art. 10,

$$= ac + bc + ad + bd - ae - be.$$

Similarly for any number of terms in either multiplier or multiplicand.

Multiplication of Multinomials by Multinomials.

13. From the preceding article is derived the following principle for multiplying a multinomial by a multinomial:

Multiply each term of the multiplicand by each term of the multiplier, and add algebraically the resulting products.

Ex. 1. Multiply $-3a+2b$ by $2a-3b$.

$$\begin{aligned}\text{We have } (-3a+2b)(2a-3b) &= (-3a)\times 2a + 2b\times 2a\\ &\quad -3a\times(-3b)+2b\times(-3b)\\ &= -6a^2+4ab+9ab-6b^2\\ &= -6a^2+13ab-6b^2.\end{aligned}$$

The work may be arranged as follows: *Write the multiplier under the multiplicand, the first partial product,* i.e., *the product of the multiplicand by the first term of the multiplier, under the multiplier, the second partial product under the first, and so on, placing like terms of the partial products in the same column.*

We have

$$\begin{array}{rrr}-3a & +\ 2b & \\ 2a & -\ 3b & \\ \hline -6a^2 & +\ 4ab & \\ & +\ 9ab & -6b^2\\ \hline -6a^2 & +13ab & -6b^2\end{array}$$

It is customary to multiply from left to right, instead of from right to left as in Arithmetic.

Ex. 2. Multiply $a^3+x^3-4a^2x-2ax^2$ by $x-3a$.

Arranging the multiplicand to descending powers of x, we have

$$\begin{array}{lllll}x^3 & -2ax^2 & -4a^2x & +a^3 & \\ x & -3a & & & \\ \hline x^4 & -2ax^3 & -4a^2x^2 & +\ a^3x & \\ & -3ax^3 & +6a^2x^2 & +12a^3x & -3a^4\\ \hline x^4 & -5ax^3 & +2a^2x^2 & +13a^3x & -3a^4\end{array}$$

Ex. 3. Multiply $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

We have

$$\begin{array}{l}
a^3 - 3a^2b + 3ab^2 - b^3 \\
a^2 - 2ab + b^2 \\
\begin{array}{r|r|r|r|r|r}
\hline
a^5 - 3 & a^4b + 3 & a^3b^2 - 1 & a^2b^3 & & \\
-2 & +6 & -6 & +2 & ab^4 & \\
 & +1 & -3 & +3 & & -b^5 \\
\hline
\end{array} \\
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5
\end{array}$$

In this work the literal parts of the first three terms of the second and third partial products were omitted, it being understood that the numerals remaining are the coefficients of the literal parts just above in the first and second partial products.

Observe that in the last example the multiplicand and multiplier, and also the product, are *homogeneous*.

Ex. 4. Multiply $2x^{m+1} - 5x^m + 7x^{m-1} - 9x^{m-2}$ by $x^{2m} - x^{2m-1}$.

We have

$$\begin{array}{l}
2x^{m+1} - 5x^m + 7x^{m-1} - 9x^{m-2} \\
x^{2m} - x^{2m-1} \\
\begin{array}{r|r|r|r|r}
\hline
2x^{3m+1} - 5 & x^{3m} + 7 & x^{3m-1} - 9 & x^{3m-2} & \\
-2 & +5 & -7 & & +9x^{3m-3} \\
\hline
\end{array} \\
2x^{3m+1} - 7x^{3m} + 12x^{3m-1} - 16x^{3m-2} + 9x^{3m-3}
\end{array}$$

EXERCISES XI.

Multiply

1. $x + 3$ by $x + 7$.
2. $2a - 7$ by $3a + 4$.
3. $-3ab + 7$ by $2ab - 5$.
4. $\frac{1}{2}a - 5$ by $\frac{1}{3}a - 8$.
5. $x + 3$ by $y + 4$.
6. $-5a + 7$ by $2b - 3$.
7. $a + b$ by $2a - 3b$.
8. $ax - by$ by $ax + 2by$.
9. $5x^2 - x$ by $1 - 2x$.
10. $-3ab + ac$ by $7ab - 5ac$.
11. $-17a^2x^7 + 12a^5x^4$ by $a^8x - 3a^2x^2$.
12. $2a^mx^{n-1} - 3a^{m-1}x^n$ by $5a^2x^n - 2a^mx^2$.
13. $\frac{2}{3}a^{m+1}b^{n-1} - \frac{1}{5}a^{n-2}b^{m+2}$ by $-\frac{1}{2}a^nb^{m+1} + a^{m+1}b^n$.
14. $x^2 - 3x + 1$ by $x - 4$.
15. $4a^2 - 6a + 9$ by $2a + 3$.
16. $1 - 2a + 4a^2 - 8a^3$ by $1 + 2a$.
17. $k^2 - k + 1$ by $k^2 + k + 1$.
18. $a^2 - ab + b^2$ by $a + b$.
19. $2a^2 - a$ by $a^2 - a + 1$.

20. $8x^3 + 12x^2y + 18xy^2 + 27y^3$ by $2x - 3y$.

21. $4a - 4a^2 + 2a^3$ by $2 + 2a + a^2$.

22. $2b^2 - 3b + 4$ by $b^2 - 2b - 3$.

23. $a^2b + 2ab^2 - 1$ by $2a^2 - ab + 1$.

24. $2x^2 + 3xy + 4y^2$ by $3x^2 - 4xy + y^2$.

25. $-1 + 3x^3 - 5x^6$ by $4x - x^4 + 2x^7$.

26. $x^3 - 2x^2 + 3x - 4$ by $4x^3 + 3x^2 + 2x + 1$.

27. $x^4 + 2x^3 + x^2 - 4x - 11$ by $x^2 - 2x + 3$.

28. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.

29. $3\frac{1}{3}a^4b + 2\frac{3}{10}a^2b^3 - 4\frac{1}{2}a^3b^2$ by $2\frac{1}{3}a^3 - 2\frac{1}{2}a^2b + ab^2$.

30. $3x^m - 2x^{m-1} + 4x^{m-2}$ by $2x^{m-1} + 4x^{m-2} - 5x^{m-3}$.

31. $3a^{p-3} + a^{p-2} - 2a^{p-1} - 4a^p$ by $2a^{p-3} + 3a^{p-4}$.

32. $3a^{n-2}x^2 - a^{n-3}x^3 + a^n$ by $a^2x^{n-1} - 3x^{n+1} - 2ax^n$.

33. $2x(a^2 + b^2)^3 - 3x^2(a^2 + b^2)^2 + 4x^3(a^2 + b^2)$
by $x(a^2 + b^2)^2 - 2(a^2 + b^2)$.

34. $(x + y)^{n+2} + 3(x + y)^{n+1} - 5(x + y)^n$
by $6(x + y)^{n+1} + 4(x + y)^n - 2(x + y)^{n-1}$.

Perform the following indicated operations:

35. $(x - 2)(x + 3)(x - 4)$.

36. $(x - 3)(x - 5)(x - 7)$.

37. $(2x - 3y)(4x + y)(x + 5y)$.

38. $(xy - 2z)(3z - 4xy)(z + 5xy)$.

39. $(2m^2 + 3m - 2)(m - 1)(2m + 3)$.

40. $(x^2 + 4x - 1)(x^2 - 2x + 1)(x + 2)$.

41. $(a^2 - a + 1)(a^2 + a + 1)(a^4 - a^2 + 1)$.

42. $(a^m + b^m)(a^n + b^n)(a^p - b^p)$.

43. $(1 + x^n + x^m)(3 - 2x^n + x^m)(5x^{n-1} - 3x^{m-1})$.

14. *The converse of the Distributive Law evidently holds;* that is,

$$ab + ac - ad = a(b + c - d), \text{ etc.}$$

E.g., $ax + bx = (a + b)x$, $2ay - 3by = (2a - 3b)y$.

15. If the coefficients of the multiplicand and multiplier, arranged to a common letter of arrangement, be literal, it is frequently desirable to unite the terms of the product which are *like in this letter of arrangement.*

Ex. Multiply $x + a$ by $x + b$.

We have
$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \underline{\phantom{x^2 +{}} bx + ab} \\ x^2 + ax + bx + ab = x^2 + (a + b)x + ab, \text{ by Art. 14.} \end{array}$$

EXERCISES XII.

Arrange the values of the following products to descending powers of x, uniting like terms in x:

1. $(x^2 + ax + b)(x + a)$.
2. $(x^3 - ax^2 + bx - c)(x - b)$.
3. $[x^2 + (a - b)x - ab](x - c)$.
4. $[x^2 - (a + b)x + ab][x^2 + (c - d)x - cd]$.
5. $(px^3 + qx^2 + rx + s)(ax^2 + bx + c)$.

Zero in Multiplication.

16. Since $N \cdot 0 = N(b - b)$, by definition of 0,
$$= Nb - Nb = 0,$$
we have $$\boldsymbol{N \cdot 0 = 0 \text{ and } 0 \cdot N = 0}.$$

That is, *a product is* 0 *if one of its factors be zero.*

17. The words *is not equal to, does not have the same value as*, etc., are frequently denoted by the symbol $\neq$.

E.g., $7 \neq 2$, read *seven is not equal to* 2.

18. It follows, conversely, from Art. 16:

If a product be 0, *one or more of its factors is* 0.

That is, if $$P \times Q = 0,$$
then either $$P = 0 \text{ and } Q \neq 0;$$
or $$Q = 0 \text{ and } P \neq 0; \text{ or } P = 0 \text{ and } Q = 0.$$

EXERCISES XIII.

1. What is the value of $2(a - b)$, when $b = a$?
2. What is the value of $(a + b)(c - d)$, when $c = d$?
3. What is the value of $(b + c)(a + b - c)$, when $c = a + b$?
4. What is the value of $(x^2 - 9)(x^4 - 7x^3 + 2x - 9)$, when $x = 3$? When $x = -3$?

If $P \times Q \times R = 0$, what can we infer,

5. When $P \neq 0$? 6. When $Q \neq 0$? 7. When $P \neq 0$ and $R \neq 0$?

For what values of x does each of the following expressions reduce to 0:

8. $x(x-2)$? 9. $(x-4)(x+7)$? 10. $(x-1)(x-a)$?
11. $(x-6)(x+8)(x^2-25)$? 12. $x(x-a)(x-b)(x-c)$?

§ 4. DIVISION.

1. One power is said to be *higher* or *lower* than another according as its exponent is *greater* or *less* than the exponent of the other.

E.g., a^4 is a higher power than a^3 or b^2, but is a lower power than a^6 or b^7.

2. Quotient of Powers of One and the Same Base.

Ex.
$$\begin{aligned} a^7 \div a^3 &= (aaaaaaa) \div (aaa) \\ &= (aaaa) \times (aaa) \div (aaa), \text{ by Assoc. Law,} \\ &= aaaa = a^4 = a^{7-3}. \end{aligned}$$

This example illustrates the following principle:

(i.) *The quotient of a higher power of a given base by a lower power of the same base, is equal to a power of that base whose exponent is the exponent of the dividend minus the exponent of the divisor;* or, stated symbolically,

$$a^m \div a^n = a^{m-n}, \text{ when } m > n.$$

For
$$\begin{aligned} a^m \div a^n &= (aaa \cdots \text{to } m \text{ factors}) \div (aaa \cdots \text{to } n \text{ factors}) \\ &= [aaa \cdots \text{to } (m-n) \text{ factors}] \times (aaa \cdots \text{to } n \text{ factors}) \\ &\qquad \div (aaa \cdots \text{to } n \text{ factors}) \\ &= aaa \cdots \text{to } (m-n) \text{ factors}, = a^{m-n}. \end{aligned}$$

(ii.) $$a^m \div a^n = 1, \text{ when } m = n.$$

E.g., $$a^2 \div a^2 = 1.$$

EXERCISES XIV.

Express each of the following quotients as a single power:

1. $2^3 \div 2$. 2. $x^5 \div x^2$. 3. $(-5)^7 \div (-5)^4$.
4. $(-6)^5 \div 6^3$. 5. $(-a)^9 \div a^4$. 6. $(ab)^5 \div (ab)^2$.
7. $(-x)^7 \div (-x)^4$. 8. $(-xy)^{11} \div (-xy)^7$. 9. $a^n \div a^3$.
10. $3^n \div 3^m$. 11. $a^{n+1} \div a$. 12. $x^{n+7} \div x^n$.
13. $b^{x+3} \div b^{x+1}$. 14. $a^n \div a^{n-1}$. 15. $a^{2n} \div a^{n-1}$.
16. $(a+b)^5 \div (a+b)^2$. 17. $(xy-1)^{2n-4} \div (xy-1)^{n-3}$.

Division of Monomials by Monomials.

3. Ex. 1. $12a \div 4 = 12 \div 4 \times a, \ = 3a.$

Ex. 2. $-27x^7 \div 3x^2 = (-27 \div 3) \times (x^7 \div x^2), \ = -9x^5.$

Ex. 3. $15a^3b^2 \div (-5ab^2) = [15 \div (-5)] \times (a^3 \div a) \times (b^2 \div b^2)$
$$= -3a^2.$$

These examples illustrate the following method:

The quotient of one monomial divided by another is the quotient of their numerical coefficients multiplied by the quotient of their literal factors.

EXERCISES XV.

Divide

1. $6a$ by 3. **2.** $12x$ by $-x$. **3.** $-15m$ by $3m$.

4. $5x^4$ by $2x$. **5.** $9x^3$ by $-3x^2$. **6.** $-11a^7$ by $-5a^2$.

7. $4ab$ by $-2a$. **8.** $6abc$ by $-3ac$. **9.** $\frac{1}{2}a^3b$ by $3a^2b$.

10. $6x^8y$ by $5x^4$. **11.** $-15a^5b^7$ by $-3ab^5$.

12. $7a^7b^{10}c^{18}$ by $-5a^4b^5c^6$. **13.** $\frac{5}{6}m^6n^7p^8$ by $-\frac{2}{3}m^2n^4p^6$.

14. $15(a+b)$ by $3(a+b)$. **15.** $25x^2(x+1)^3$ by $-5x(x+1)^2$.

16. $10a^{2n}b^5$ by $-5a^nb^3$. **17.** $-27x^{n+1}y^{3m}$ by $-9xy^{2m}$.

18. $x^{2n-1}y^{3m+2}$ by $x^{n+1}y^{2m-3}$. **19.** $a^{n-1}b^{n-2}$ by $a^{n-3}b^{n-4}$.

Simplify

20. $a^3x^5 \div (-ax^3) \times 2axy$. **21.** $35x^2y^3z \times 2xz^3 \div (7x^2y^2z^2)$.

22. $a^{2n-1}b^{m+1}c^{m+n} \div a^nb^mc^m \div a^{n-2}bc^{n-3}$.

23. $6x^{m+1}y^{n-1} \div (-x^{m-1}y^{m-n}) \times (3x^2y^2z^2)$.

The Distributive Law for Division.

4. If the indicated operation within the parentheses in the quotient $(8+6) \div 2$ be first performed, we have

$$(8+6) \div 2 = 14 \div 2 = 7.$$

But if each term within the parentheses be first divided by 2 and the resulting quotients be then added, we have

$$8 \div 2 + 6 \div 2 = 4 + 3 = 7, \text{ as above.}$$

Therefore $(8+6) \div 2 = 8 \div 2 + 6 \div 2.$

This example illustrates the following principle:

Distributive Law. — *The quotient of a multinomial by a monomial is obtained by dividing each term of the multinomial by the monomial and adding algebraically the resulting quotients;* that is,

$$(a + b - c) \div d = a \div d + b \div d - c \div d.$$

For, since $\div\, d \times d = \div\, 1$, we have

$$\begin{aligned}(a + b - c) \div d &= (a \div d \times d + b \div d \times d - c \div d \times d) \div d \\ &= (a \div d + b \div d - c \div d) \times d \div d, \qquad \text{by § 3, Art. 14,} \\ &= a \div d + b \div d - c \div d, \text{ since } \times\, d \div d = \times\, 1.\end{aligned}$$

5. It follows, conversely, from the Distributive Law that

$$a \div d + b \div d - c \div d = (a + b - c) \div d.$$

Zero in Division.

6. Since $0 \div N = (a - a) \div N$, by definition of 0,

$$= a \div N - a \div N = 0.$$

We have $0 \div N = 0$, when $N \neq 0$.

Observe that this relation is proved only when $N \neq 0$.

7. It follows, conversely, from Art. 6:

If a quotient be 0, *the dividend is* 0.

That is, if $M \div N = 0$, then $M = 0$.

Division of a Multinomial by a Monomial.

8. The division of a multinomial by a monomial is a direct application of the Distributive Law.

Ex. 1. Divide $6x^2 - 12x$ by $3x$.

We have $(6x^2 - 12x) \div 3x = 6x^2 \div 3x - 12x \div 3x$

$$= 2x - 4.$$

Ex. 2. Divide $-105a^3b^2 - 75a^2b^3 + 27a^2b^4$ by $-15a^2b$.

We have $(-105a^3b^2 - 75a^2b^3 + 27a^2b^4) \div (-15a^2b)$

$$\begin{aligned}&= (-105a^3b^2) \div (-15a^2b) - 75a^2b^3 \div (-15a^2b) \\ &\qquad + 27a^2b^4 \div (-15a^2b) \\ &= 7ab + 5b^2 - \tfrac{9}{5}b^3.\end{aligned}$$

EXERCISES XVI.

Divide

1. $5 + 10a$ by 5. **2.** $4a + 8b$ by -4.

3. $ax + bx$ by x. **4.** $3a^2 - 6ab$ by $-3a$.

5. $21a^2b - 14ab^2$ by $-7ab$. **6.** $8am^2 - 2a^2m + 4a^3m^2$ by $2am$.

7. $25(a+b)^3 - 2a(a+b)$ by $5(a+b)$.

8. $2(x-y)^3 - 2a(x-y)^4 - 6(x-y)^6$ by $2(x-y)^2$.

Simplify

9. $2a^3 - (a^3 - 3a) \div a$.

10. $(6x - 4x^2) \div 2x - (-2x^2y + 3xy) \div xy$.

11. $(ab - a^2b + 3a^3b) \div ab - (4a^3 - 4a^2) \div 2a$.

Divide $9a^2x^6 - 6a^3x^4 + 12a^5x^3$ by

12. $3a^2$. **13.** $-3x^3$. **14.** ax^2. **15.** $-\frac{3}{2}a^2x^3$.

Divide $35a^3b^2c^4 - 21a^4b^3c^3 + 14a^5b^4c^3$ by

16. $7a^3$. **17.** $-3a^3b^2$. **18.** $-5a^2bc^3$. **19.** $\frac{3}{4}a^2b^2c^2$.

Divide $15x^{2n+1}y^5 - 12x^{2n+3}y^3 - 18x^{2n+5}y^4$ by

20. $3x^n$. **21.** $-5x^{n+1}y^2$. **22.** $-3x^{2n+1}y$. **23.** $\frac{1}{2}x^{2n-5}y^3$.

24. Prove that the sum, or the difference, of two even numbers is exactly divisible by 2, and is therefore an even number.

25. Prove that the sum, or the difference, of two odd numbers is even.

26. Prove that the sum, or the difference, of an even and an odd number is odd.

Division of a Multinomial by a Multinomial.

9. The division of one multinomial by another is performed in a way similar to that of dividing one number by another in Arithmetic.

Ex. Divide 125 by 5.

We have

$$\begin{array}{r|l} 125 & \underline{\;5\qquad\;} \\ \underline{100} & 20 + 5, \; = 25 \\ 25 & \\ \underline{25} & \end{array} \qquad \text{or, omitting ciphers,} \qquad \begin{array}{r|l} 125 & \underline{\;5\;} \\ \underline{10} & 25 \\ 25 & \\ \underline{25} & \end{array}$$

The work is equivalent to the following:

$$125 \div 5 = 20 + (125 - 20 \times 5) \div 5 = 20 + 25 \div 5 = 25.$$

This example illustrates the following principle:

The quotient of dividing one number **(dividend)** *by another* **(divisor)** *is equal to any number whatever* **(partial quotient),** *plus the quotient of dividing the dividend minus the partial quotient times the divisor, by the divisor.*

If D be the given *dividend*, d the given *divisor*, and q *any assumed number*, the principle enunciated above, stated symbolically, is:

$$\boldsymbol{D \div d = q + (D - qd) \div d.}$$

Although the partial quotient may be *any number whatever*, yet in practice we should take the greatest number whose product by the divisor is equal to or less than the dividend.

A quotient consisting of more than one figure is obtained by successive applications of the same principle.

We have

$$\begin{aligned}[q + (D - qd) \div d] &= [q + (D - qd) \div d] \times d \div d, \text{ since } \times d \div d = \times 1,\\ &= [qd + (D - qd) \div d \times d] \div d, \text{ by the Distr. Law,}\\ &= [qd + (D - qd)] \div d, \text{ since } \div d \times d = \div 1,\\ &= D \div d, \text{ since } qd - qd = 0.\end{aligned}$$

10. The principle of Art. 9 evidently holds when the dividend, D, and the divisor, d, are algebraic expressions.

Ex. Divide $x^2 + 3x + 2$ by $x + 1$.

We have

$$\begin{aligned}(x^2+3x+2) \div (x+1) &= x + [(x^2+3x+2) - x(x+1)] \div (x+1) &(1)\\ &= x + (x^2+3x+2-x^2-x) \div (x+1) &(2)\\ &= x + (2x+2) \div (x+1) &(3)\\ &= x+2+[(2x+2)-2(x+1)] \div (x+1) &(4)\\ &= x+2+0 \div (x+1)\\ &= x+2, \text{ since } 0 \div (x+1) = 0.\end{aligned}$$

We take the quotient of the term containing the highest power of x in the dividend by the term containing the highest power of x in the divisor as the partial quotient at each step.

The work may be arranged more conveniently thus:

$$\begin{array}{l|l} x^2+3x+2 & x+1 \\ \cline{2-2} & x+2,\ \text{quotient.} \end{array}$$

$\underline{x^2 + \quad x\qquad}$ ⋯ $x(x+1)$ to be subtracted from x^2+3x+2; see (1) and (2) above.

$\qquad 2x+2$ ⋯ Remainder to be divided by $x+1$; see (3) above.

$\qquad \underline{2x+2}$ ⋯ $2(x+1)$ to be subtracted from $2x+2$; see (4).

11. The method of applying the principle of Art. 9 to the division of multinomials, as illustrated by this example, may be stated as follows:

Arrange the dividend and divisor to ascending or descending powers of some common letter, the letter of arrangement.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the divisor by this first term of the quotient, and subtract the resulting product from the dividend.

Divide the first term of the remainder by the first term of the divisor, and write the result as the second term of the quotient.

Multiply the divisor by this second term of the quotient, and subtract the product from the remainder previously obtained. Proceed with the second remainder and all subsequent remainders, in like manner, until a remainder zero is obtained, or until the highest power of the letter of arrangement in the remainder is less than the highest power of that letter in the divisor.

In the first case the division is exact; in the second case the quotient at this stage of the work is called the quotient of the division, and the remainder the remainder of the division.

Ex. 1. Divide

$$a^3b - 15b^4 + 19ab^3 + a^4 - 8a^2b^2 \text{ by } a^2 - 5b^2 + 3ab.$$

Arranging dividend and divisor to descending powers of a, we have

$$\begin{array}{l|l}
a^4 + a^3b - 8a^2b^2 + 19ab^3 - 15b^4 & a^2 + 3ab - 5b^2 \\ \cline{2-2}
\underline{a^4 + 3a^3b - 5a^2b^2} & a^2 - 2ab + 3b^2 \\
\quad -2a^3b - 3a^2b^2 + 19ab^3 & \\
\quad \underline{-2a^3b - 6a^2b^2 + 10ab^3} & \\
\qquad\qquad 3a^2b^2 + 9ab^3 - 15b^4 & \\
\qquad\qquad \underline{3a^2b^2 + 9ab^3 - 15b^4} &
\end{array}$$

Ex. 2. Divide $8x^3 - y^3$ by $2xy + 4x^2 + y^2$.

Arranging the divisor to descending powers of x, we have

$$\begin{array}{l|l}
8x^3 - y^3 & 4x^2 + 2xy + y^2 \\ \cline{2-2}
\underline{8x^3 + 4x^2y + 2xy^2} & 2x - y \\
\quad -4x^2y - 2xy^2 - y^3 & \\
\quad \underline{-4x^2y - 2xy^2 - y^3} &
\end{array}$$

Observe that the remainder after the first partial division is arranged to descending powers of x.

Ex. 3. Divide $12a^{n+1} + 8a^n - 45a^{n-1} + 25a^{n-2}$ by $6a - 5$.

We have

$$\begin{array}{l|l}
12a^{n+1} + 8a^n - 45a^{n-1} + 25a^{n-2} & 6a - 5 \\ \cline{2-2}
\underline{12a^{n+1} - 10a^n} & 2a^n + 3a^{n-1} - 5a^{n-2} \\
\qquad 18a^n - 45a^{n-1} & \\
\qquad \underline{18a^n - 15a^{n-1}} & \\
\qquad\qquad -30a^{n-1} + 25a^{n-2} & \\
\qquad\qquad \underline{-30a^{n-1} + 25a^{n-2}} &
\end{array}$$

Ex. 4. Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $x^2 + (a + b)x + ab$.

We have

$$\begin{array}{l|l}
x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc & x^2 + (a + b)x + ab \\ \cline{2-2}
\underline{x^3 + (a + b\phantom{{}+c})x^2 + ab\,x} & x + c \\
\qquad\qquad cx^2 + (ac + bc)x + abc & \\
\qquad\qquad \underline{cx^2 + (ac + bc)x + abc} &
\end{array}$$

EXERCISES XVII.

Find the values of the following indicated divisions:

1. $(x^2 + 2x + 1) \div (x + 1)$. 2. $(x^2 + 11x + 30) \div (x + 5)$.
3. $(x^2 - x - 90) \div (x + 9)$. 4. $(x^2 - 5x + 6) \div (x - 3)$.
5. $(4x^2 - 12x + 9) \div (2x - 3)$. 6. $(2m^2 - 3m + 1) \div (m - 1)$.
7. $(2a^2 + a - 6) \div (2a - 3)$. 8. $(3x^2 - 13x - 10) \div (3x + 2)$.

9. $(6x^4 - 10 - 11x^2) \div (2x^2 - 5)$.
10. $(2x^2 + 6a^2 + 7ax) \div (2x + 3a)$.
11. $(a^2 - 2ab + b^2) \div (a - b)$.
12. $(35x^2 + xy - 88y^2) \div (7x - 11y)$.
13. $(x^2 + 5\frac{2}{3}xy + 3\frac{1}{3}y^2) \div (x + 5y)$.
14. $(\frac{1}{9}a^2 + \frac{2}{5}ab + \frac{4}{25}b^2) \div (\frac{1}{3}a + \frac{4}{5}b)$.
15. $(a^2 - 18axy - 243x^2y^2) \div (a + 9xy)$.
16. $(8x^2y^2 - 65xyz^2 - 63z^4) \div (xy - 9z^2)$.
17. $(6n^3 - 7n^2x + 2nx^2) \div (-x + 2n)$.
18. $(x^4y + 6x^5 - 2x^3y^2) \div (3x^2 + 2xy)$.
19. $(-19a^2x^2 + 3x^4 + \frac{7}{2}ax^3) \div (\frac{1}{2}x - a)$.
20. $(4x^3 - 3x^2 - 24x - 9) \div (x - 3)$.
21. $(3x^3 - 13x^2 + 23x - 21) \div (3x - 7)$.
22. $(3x^4 - 3x^3 - 2x^2 - x - 1) \div (3x^2 + 1)$.
23. $(a^3 - 3a^2b + 3ab^2 - b^3) \div (a - b)$.
24. $(a^6 - 6a^4 + 9a^2 - 4) \div (a^2 - 1)$.
25. $(21a^6b + 20b^4 - 22a^2b^3 - 29a^4b^2) \div (3a^2b - 5b^2)$.
26. $(4x^4y^5 - \frac{1}{2}x^2y^6 + 12x^8y^3 - 11x^6y^4) \div (4x^3y^2 - xy^3)$.
27. $(x^3 + 8x^2 + 9x - 18) \div (x^2 + 5x - 6)$.
28. $(x^4 + x^3 - 4x^2 + 5x - 3) \div (x^2 + 2x - 3)$.
29. $(6x^4 - x^3 - 11x^2 - 10x - 2) \div (2x^2 - 3x - 1)$.

30. $(x^3 - 1) \div (x^2 + x + 1)$. 31. $(a^3 + 8) \div (a^2 - 2a + 4)$.
32. $(125x^6 - 64y^3) \div (5x^2 - 4y)$. 33. $(a^5x^5 + y^5) \div (ax + y)$.
34. $(x^4 + x^2 + 1) \div (x^2 - x + 1)$. 35. $(a^4x^5 + 64x) \div (4ax + a^2x^2 + 8)$.

36. $(24x^3 + 25x - x^5) \div (5 + x + x^3 + 5x^2)$.
37. $(4a^4 - 25c^4 - 30b^2c^2 - 9b^4) \div (2a^2 + 5c^2 + 3b^2)$.
38. $(27x^4 - 6c^2x^2 + \frac{1}{3}c^4) \div (c^2 - 6cx + 9x^2)$.
39. $(8a^3n^3 + 32a^6 + \frac{1}{2}n^6) \div (4an + n^2 + 4a^2)$.

40. $(16a^4b^2 + 9a^2b^4 - 12a^3b^3 - 8a^5b + 3a^6) \div (a^4 + 3a^2b^2 - 2a^3b)$.

41. $(28a^5c - 26a^3c^3 - 13a^4c^2 + 15a^2c^4) \div (2a^2c^2 + 7a^3c - 5ac^3)$.

42. $(81z^8 - 90b^4z^4 + 81b^6z^2 - 20b^8) \div (9z^4 + 9b^2z^2 - 5b^4)$.

43. $(x^3 + y^3 + 3xy - 1) \div (x + y - 1)$.

44. $(a^3 + b^3 + c^3 - 3abc) \div (a + b + c)$.

45. $(a^2 + 2ab + b^2 - x^2 + 4xy - 4y^2) \div (a + b - x + 2y)$.

46. $(a^2 + 2ac - b^2 - 2bd + c^2 - d^2) \div (a + c - b - d)$.

47. $(32a^5 + b^5) \div (16a^4 - 8a^3b + 4a^2b^2 - 2ab^3 + b^4)$.

48. $(81x^8 - 16y^8) \div (27x^6 + 18x^4y^2 + 12x^2y^4 + 8y^6)$.

49. $(\frac{15}{2}a^2 - \frac{113}{6}ab + 9ac + 2b^2 - bc) \div (\frac{5}{4}a - 3b + \frac{3}{2}c)$.

50. $(28x^2 - 43\frac{3}{4}y^2 + 140yz - 112z^2) \div (7x + 8\frac{3}{4}y - 14z)$.

51. $(\frac{3}{8}a^2b + 6acd - \frac{8}{3}bc^2 + 16c^2d - \frac{1}{2}abd + \frac{4}{3}bcd - 8cd^2) \div (\frac{3}{2}a + 4c - 2d)$.

52. $(\frac{2}{3}a^4x - 1\frac{61}{75}a^3x^2 + 1\frac{3}{5}a^2x^3 + \frac{3}{10}ax^4 - x^5) \div (\frac{2}{3}a^3 - \frac{4}{5}a^2x + \frac{1}{2}x^3)$.

Find the values of the following indicated divisions:

53. $[(b + c)x^2 - bcx + x^3 - bc(b + c)] \div (x^2 - bc)$.

54. $[x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc] \div (x + b)$.

55. $[x^3 + (a + b - c)x^2 + (ab - ac - bc)x - abc] \div (x - c)$.

56. $[abc - b^2(a + c) + a^2(b + c) + c^2(a + b)] \div (ab + ac + bc)$.

57. $[a(a - 1)x^3 + (a^3 + 2a - 2)x^2 + (3a^2 - a^3)x - a^4] \div (ax^2 + 2x - a^2)$.

58. $[x^5 - (1 + m)x^4 + (1 + m + n)x^3 - (m + n + p)x^2 + (p + n)x - p]$
$\div [x^2 - (x - 1)]$.

59. $[(10a^2 + 29 - 34a)x + (5 - 2a^2 + 3a^3 - 8a)x^3 + 8a^2 + 21 - 26a$
$+ (17 - 22a + 4a^3)x^2] \div [(a - 2)x + (a^2 - 1 + a)x^2 + 2a - 3]$.

Find the values of the following indicated divisions:

60. $(6x^{3n} - 25x^{2n} + 27x^n - 5) \div (2x^n - 5)$.

61. $(6x^{5n} - 11x^{4n} + 23x^{3n} + 13x^{2n} - 3x^n + 2) \div (3x^n + 2)$.

62. $(6x^{2n+1} - 29x^{2n} + 43x^{2n-1} - 20x^{2n-2}) \div (2x^n - 5x^{n-1})$.

63. $(1 + a^{6x} - 2a^{3x}) \div (3a^{2x} + 2a^{3x} + 2a^x + a^{4x} + 1)$.

64. $[2a^2(b + c)^{2n} - \frac{1}{2}] \div [a(b + c)^n + \frac{1}{2}]$.

65. $[8(x - y)^{3n} - x^3] \div [4(x - y)^{2n} + 2x(x - y)^n + x^2]$.

12. In the equation $D \div d = q + (D - qd) \div d$, $D - qd$ is the remainder at any stage of the work, and q is the corresponding partial quotient. If, for brevity, we let R stand for the remainder at any stage, we have,

$$D \div d = q + R \div d. \tag{1}$$

That is, ***the result of dividing one number by another is equal to the partial quotient at any stage, plus the remainder at this stage divided by the given divisor.***

E.g., $29 \div 6 = 4 + 5 \div 6 = 4 + \frac{5}{6}$;

$$(x^2 - x + 2) \div (x + 1) = (x - 2) + 4 \div (x + 1).$$

13. If both members of the equation

$$D \div d = q + R \div d$$

be multiplied by d, we have, by Ch. I., § 1, Art. 15 (III.),

$$\begin{aligned} D \div d \times d &= (q + R \div d)\, d \\ &= qd + R \div d \times d \\ &= qd + R, \text{ since } \div d \times d = \div 1. \end{aligned}$$

Therefore, $D = qd + R$.

That is, ***the dividend is equal to the product of the quotient at any stage by the divisor, plus the remainder at this stage.***

E.g., $29 = 4 \times 6 + 5$, and $x^2 - x + 2 = (x - 2)(x + 1) + 4$.

EXERCISES XVIII.

Find the remainder of each of the following indicated divisions, and verify the work by applying the principle of Art. 13:

1. $(x^2 - 7x + 11) \div (x - 2)$.
2. $(3x^2 + 5x - 9) \div (x - 4)$.
3. $(x^3 - 17x^2 + 15x - 13) \div (2x - 5)$.
4. $(5x^5 - 7x^2 + 2x - 1) \div (x^2 - 7x + 3)$.
5. $(6n^5x^3 + 12n^3x^9 - 14n^4x^6 + n^6 - 1) \div (2x^3 - n)$.
6. $(12b^5 + 8b^2c^3 - 2b^4c - 4bc^4 - 38b^3c^2) \div -(2c^2 + 6bc - 4b^2)$.
7. $(4c^{2n}x^{3n} - 13c^{3n}x^{2n} + 14c^{4n}x^n - 2c^{5n}) \div (c^nx^{2n} - 2c^{2n}x^n + c^{3n})$.

Infinites.

14. The following considerations lead to an important mathematical concept.

Observe that the quotients

$$\begin{aligned} 1 \div (1 - .9) &= 1 \div .1 &= 10, \\ 1 \div (1 - .99) &= 1 \div .01 &= 100, \\ 1 \div (1 - .999) &= 1 \div .001 &= 1000, \text{ etc.,} \end{aligned}$$

increase as the divisors decrease, the dividend remaining the same.

If the divisor be still further decreased, the dividend remaining the same, the quotient will be still further increased.

Thus, $1 \div (1 - .999999999) = 1 \div .000000001 = 1000000000$, etc.

It is evident that, by taking the divisor sufficiently small (and positive), we can make the quotient as great as we please. If the divisor become less than any assigned number, however small, the quotient will become greater than any assigned number, however great. That is,

If the dividend be positive, and remain the same, as the divisor decreases below any assigned positive number, however small, i.e., *becomes more and more nearly equal to* 0, *the quotient increases beyond any assigned positive number, however great.*

The symbol, $+\infty$, read a **Positive Infinite Number**, or a **Positive Infinite**, is used as an abbreviation for the words, *a number greater than any assigned positive number, however great.*

The principle enunciated above can be expressed symbolically thus:

$$+N \div 0 = +\infty. \qquad (1)$$

15. It is important to observe that the symbol, $+\infty$, does not stand for one definite number. It stands for *any* number which is greater than any assigned positive number, however great, *but which can be still further increased.* Therefore one infinite number can be greater or less than another infinite number.

Likewise, equation (1), Art. 14, is to be understood only as expressing the fact that, as the divisor becomes more and more nearly equal to 0, the quotient increases beyond any assigned positive number, however great.

16. We can also arrive at the conception of a positive infinite number by taking the dividend and the divisor both negative. Thus,

$$-1 \div (1 - 1.1) = -1 \div (-.1) = 10,$$
$$-1 \div (1 - 1.0001) = -1 \div (-.0001) = 10000, \text{ etc.}$$

17. In a similar manner, we gain the conception of a **Negative Infinite Number**, or a **Negative Infinite**.

Thus, $-1 \div (1 - .9) = -1 \div .1 = -10,$

$$-1 \div (1 - .9999) = -1 \div .0001 = -10000, \text{ etc.}$$

We therefore have $-N \div 0 = -\infty.$

18. The numbers which we have hitherto used in this book are, for the sake of distinction, called *finite* numbers. In subsequent work we shall assume that the numbers involved are finite, unless the contrary is expressly stated.

19. Observe that the quotients

$$1 \div 10 = .1,\ \ 1 \div 100 = .01,\ \ 1 \div 1000000 = .000001, \text{ etc.};$$

decrease as the divisors increase. It is evident that if the divisor become greater than any assigned number, however great, the quotient will become less than any assigned number, however small. That is,

If the dividend remain the same, as the absolute value of the divisor increases beyond any assigned number, however great, the quotient de-

reases in absolute value below any assigned number, however small, i.e., *ecomes more and more nearly equal to* 0. Or, stated symbolically,

$$M \div (\pm \infty) = 0.$$

20. If $M \div N = 0$, and $N \neq \infty$, then by Art. 7, $M = 0$.

21. The consideration of other relations which involve 0 and ∞ is deferred. It is important to notice that the relation

$$0 \times a = 0$$

of § 3, Art. 16, was proved only for the case in which a is *finite*.

EXERCISES XIX.

MISCELLANEOUS EXAMPLES.

Given

$$a = 2x^4 - 3x^3 + 5x^2 - 7, \qquad b = x^3 + 11x^2 - 3x + 5,$$
$$c = 7x^4 + 5x^3 - 9x^2 - 6x, \qquad d = 5x^4 - 7x^2 + 3x - 1,$$

find the values of

1. $a + b + c - d$. **2.** $a - b + c - d$. **3.** $a - b - c + d$.

4. $a + b - 7(c - d)$. **5.** $b - c + 3x(a + d)$. **6.** $d - c - 5x^2(a - b)$.

7. $(a + b)(c - d)$. **8.** $(a - d)(b - c)$. **9.** $(a - b + c)d$.

Multiply:

10. $4a^4x^{3n+3} - \frac{1}{5}a^2x^{2n+1} + 10x^{n-1}$ by $\frac{3}{20}a^2x^{2n-1} + 7\frac{1}{2}x^{n-3}$.

11. $4a^{m+1}b^2 + a^{m-2}b - 2a^{m+4}b^3$ by $3a^mb^2 - a^{m+3}b^3 - 5a^{m+6}b^4$.

12. $5a^{n+3r}b^{r-1} - 2a^{n-r}b^{r+1} + 3a^{n+3r}b^{r-2} + a^{n+r}b^r$

by $a^{n+r}b^r + 4a^{n-5r}b^{r+3} - 2a^{n-3r}b^{r+2}$.

13. $x^4(x^2 + 2)^{n-3} + 2x^2(x^2 + 2)^{2n-1} + 4(x^2 + 2)^{3n+1}$

by $x^7(x^2 + 2)^{n-5} - 4x^3(x^2 + 2)^{3n-1} + 8x(x^2 + 2)^{4n+1}$.

Divide:

14. $2x^{10} - .075x^5 + 9.65x^7 - 1.05x^6 - 19.25x^8 + 8.5x^9$

by $2.5x^6 - 3x^5 + .5x^7 - .15x^4$.

15. $6a^{4n+1}b^{m+6} - \frac{5}{2}a^{4n}b^{m+5} + \frac{1}{6}a^{4n-1}b^{m+4} + \frac{9}{4}a^{4n-2}b^{m+3} - \frac{3}{16}a^{4n-3}b^{m+2}$

by $3a^{2n+1}b^{m+1} - \frac{1}{4}a^{2n}b^m$.

16. $15a^{3n-2m-4}b^{2p+7} + 14a^{6n-m-4}b^{4p+4} - \frac{24}{5}a^{9n-4}b^{6p+1}$

by $5a^{2n-4}b^{3-p} + 6a^{5n+m-4}b^p$.

What is the value of $(x + 1)(x + 2)(x + 3)\cdots(x + n)$, when

17. $x = 1,\ n = 3$? **18.** $x = -2,\ n = 5$? **19.** $x = 8,\ n = 4$?

20. What is the value of

$(n - 5)^{n-1}(n - 4)^{n-2}(n - 5)^{n-3} - (n - 2)^{n-4}(n - 1)^{n-5}$, when $n = 6$?

21. What is the value of $s(s - a)(s - b)(s - c)$, when

$$s = \tfrac{1}{2}(a + b + c),\ a = 5,\ b = 6,\ c = 9?$$

CHAPTER IV.

INTEGRAL ALGEBRAIC EQUATIONS.

An equation has been defined (Ch. I., § 1, Art. 12) as a statement that two numbers or expressions are equal.

We must now distinguish between two kinds of equations.

§ 1. IDENTICAL EQUATIONS.

1. Examples of the one kind are:

$$(a+b)(a-b)=a^2-b^2. \qquad (1)$$

$$(a^2-b^2)\div(a-b)=(a+b)^2\div(a+b). \qquad (2)$$

The first member of (1) is reduced to the second member by performing the indicated multiplication. Both members of (2) are reduced to the common form, $a+b$, by performing the indicated divisions.

2. An **Identical Equation,** or simply an **Identity,** is an equation one of whose members can be reduced to the other, or both of whose members can be reduced to a common form, by performing the indicated operations.

3. Notice that identical equations are true for all values that may be substituted for the literal numbers involved.

E.g., if $a=5$ and $b=3$, equation (1) becomes

$$8\times 2=25-9, \text{ or } 16=16;$$

and equation (2) becomes

$$16\div 2=64\div 8, \text{ or } 8=8.$$

We need not further discuss identical equations, since we have constantly dealt with them in the preceding chapters.

§ 2. CONDITIONAL EQUATIONS.

1. Examples of the second kind are:

$x+1=3.$ (1) $\qquad x^2-1=8.$ (2) $\qquad x+y=5.$ (3)

The first member of (1) reduces to the second member, when $x = 2$. It seems evident, and we shall later prove, that $x + 1$ reduces to 3 *only* when $x = 2$.

The first member of (2) reduces to the second member, when $x = +3$ and when $x = -3$. We shall later prove that $x^2 - 1$ reduces to 8 *only* when $x = +3$ or -3.

The first member of (3) reduces to the second member, when $x = 1$ and $y = 4$, when $x = -3$ and $y = 8$; but *not* when $x = 5$ and $y = 6$, when $x = -4$ and $y = 8$. Therefore, equation (3) is true for many pairs of values of x and y, but not for all pairs of values chosen at random.

2. Such equations *impose conditions* upon the values of the literal numbers involved. Thus, equation (1) imposes the condition that if 1 be added to the value of x, the sum will be 3.

A **Conditional Equation** is an equation one of whose members can be reduced to the other only for certain definite values of one or more letters contained in it.

Whenever the word *equation* is used in subsequent work, we shall understand by it a *conditional equation*, unless the contrary is expressly stated.

3. The **Unknown Numbers** of an equation are the numbers whose values are fixed or determined by the equation.

The **Known Numbers** of an equation are the numbers whose values are given or known.

In the equation $$x^2 - 1 = 8$$

the unknown number is x, and the known numbers are 1 and 8.

In the equation $$x + y = 3$$

the unknown numbers are x and y; the known number is 3.

The unknown numbers are usually represented by the final letters of the alphabet, x, y, z, etc., as in the above examples.

4. An **Integral Algebraic Equation** is an equation whose members are integral algebraic expressions in the unknown number or numbers.

The known numbers may enter in any way whatever.

E.g., $3x^2 - 4 = 2x$, and $\frac{3}{8}x + 5y = \frac{1}{8}$, are integral equations.

5. The **Degree** of an integral equation is the degree of its term of highest degree in the unknown number or numbers.

6. A **Linear** or **Simple Equation** is an equation of the *first* degree.

E.g., $x + 1 = 6$ is a linear equation in one unknown number; $2x + 3y = 5$ is a linear equation in two unknown numbers.

7. A **Solution** of an equation is a value of the unknown number, or a set of values of the unknown numbers, which if substituted in the equation, converts it into an identity.

E.g., 2 is a solution of the equation $x + 1 = 3$,
since, when substituted for x in the equation, it converts the equation into the identity $2 + 1 = 3$.

Solutions of the equation $x^2 - 1 = 8$ are $+3$ and -3.

The set of values 1 and 2, of x and y, respectively, is a solution of the equation $x + y = 3$.

8. To **Solve** an equation is to find its solution.

The process of solving an equation is also frequently called *the solution* of the equation.

An equation is said *to be satisfied by its solution*, or *the solution is said to satisfy the equation*, since it converts the equation into an identity.

9. When the equation contains only one unknown number, a solution is frequently called a **Root** of the equation.

E.g., 3 and -3 are roots of the equation $x^2 - 1 = 8$.

§ 3. EQUIVALENT EQUATIONS.

1. We shall now give some principles upon which the solution of integral equations depends. But it is to be kept in mind that the final test of the correctness of a solution, no matter how obtained, *is that it shall satisfy the given equation.*

Consider the equation $\frac{2}{3}x - 5 = 1$. (1)

Adding 5 to both members, by Ch. I., § 1, Art. 15 (i.),
we have $\frac{2}{3}x - 5 + 5 = 1 + 5$, or $\frac{2}{3}x = 6$. (2)

Multiplying both members of (2) by 3, by Ch. I., § 1, Art. 15 (iii.),

we have $$2x = 18. \tag{3}$$

Dividing both members of (3) by 2, by Ch. I., § 1, Art. 15 (iv.),

we have $$x = 9. \tag{4}$$

It is evident that equations (1), (2), (3), and (4) are satisfied by the same root 9.

2. *Two equations are* **equivalent** *when every solution of the first is a solution of the second, and every solution of the second is a solution of the first.*

E.g., equations (1), (2), (3), and (4) of Art. 1.

3. The methods of solving integral equations depend upon principles which enable us to change a given equation into an equivalent equation whose solution is more easily obtained than that of the given one. This process is called *transforming the equation*, or *the transformation of the equation.*

Fundamental Principles for solving Integral Equations.

4. In the principles of equivalent equations which we shall now prove, the solutions are limited to finite values.

5. The transformations made in the example of Art. 1 illustrate the following principles:

(i.) **Addition and Subtraction.** — *If the same number or expression be added to, or subtracted from, both members of an equation, the derived equation will be equivalent to the given one.*

(ii.) **Multiplication.** — *If both members of an equation be multiplied by one and the same number, not* 0, *or by an expression which does not contain the unknown number or numbers, the derived equation will be equivalent to the given one.*

(iii.) **Division.** — *If both members of an equation be divided by one and the same number, not* 0, *or by an expression which does not contain the unknown number or numbers, the derived equation will be equivalent to the given one.*

(i.) Let
$$P = Q$$
be the given equation, and N be any number or expression. Then the equation
$$P \pm N = Q \pm N,$$
wherein the upper signs, +, go together and the lower signs, −, go together, is equivalent to the given one.

For any solution of the given equation makes P equal to Q. Therefore, by Ch. I., § 1, Art. 15 (i.) and (ii.), that solution makes $P \pm N$ equal to $Q \pm N$, and hence is a solution of the derived equation. Consequently, no solution is lost by the transformation.

But the given equation is obtained from the derived equation by subtracting the number or expression which was added, or by adding the number or expression which was subtracted, in forming the derived equation. Therefore any solution of the derived equation is a solution of the given equation, and no solution is gained by the transformation. Consequently, the two equations are equivalent.

(ii.) It is more convenient to prove this principle when all the terms of the equation are in the same member, say the first. The latter equation is, as we have seen, equivalent to the given one. Then any solution must reduce the first member to 0.

Let
$$P = 0$$
be the given equation, and N be any number, not 0, or any expression which does not contain the unknown number or numbers. Then the equation
$$N \cdot P = N \cdot 0 = 0$$
is equivalent to the given one.

For any solution of the given equation must reduce P to 0, and, therefore, by Ch. III., § 3, Art. 16, must also reduce $N \cdot P$ to 0. Hence it is also a solution of the derived equation. That is, no solution is lost by the transformation.

Any solution of the derived equation must reduce $N \cdot P$ to 0. But N is not 0, and, since it does not contain the unknown number or numbers, it cannot reduce to 0 for any value of the unknown number or numbers. Consequently, by Ch. III., § 3, Art. 18, any solution of the derived equation must reduce P to 0, and hence is a solution of the given equation. That is, no solution is gained by the transformation. Consequently, the two equations are equivalent. In a similar way (iii.) is proved.

6. It is important to notice that the solution of an equation does not rest simply on the principles used in Art. 1. For, by these principles we should be permitted to multiply both members of the equation by 0, or to multiply or divide both members by an expression which contains the unknown numbers.

If the multiplier were 0, any value of the unknown number would be a solution of the derived equation, but not of the given equation.

E.g., $2x-6=0$ has the root 3, while $(2x-6)\times 0=0$ is evidently satisfied by 1, 2, 3, 4, etc., without end.

If the multiplier contain the unknown number or numbers, values of the unknown number or numbers will reduce the multiplier to 0, and therefore the first member of the derived equation to 0, without reducing the first member of the given equation to 0.

E.g., $2x-6=0$ has the root 3, while $(2x-6)(x-2)=0$ is satisfied not only by 3, since $(6-6)\times 1=0\times 1=0$, but also by 2, since $(4-6)(2-2)=(-2)\times 0=0$.

But 2 is not a solution of the given equation. That is, in multiplying both members of the given equation by $x-2$, we have gained a root 2. The derived equation is, therefore, not equivalent to the given one.

If the divisor be an expression which contains the unknown number or numbers, one or more solutions are lost.

E.g., the equation $x^2-1=2(x+1)$ is satisfied by the two roots -1 and 3.

Dividing members by $x+1$, we obtain $x-1=2$.

This equation is satisfied by 3, but not by -1.

The derived equation is, therefore, not equivalent to the given one.

Applications.

7. The following applications of the preceding principles will simplify the work of solving an equation.

(i.) *Any term may be transferred from one member of an equation to the other, if its sign be reversed from* $+$ *to* $-$, *or from* $-$ *to* $+$.

E.g., $2x-4=x+1$ and $2x-x=1+4$ are equivalent equations. This step is equivalent to adding 4 to, and subtracting x from, both members of the given equation.

(ii.) *The same term, or equal terms, may be dropped from both members of an equation.*

E.g., $2x - 3 + 8 = x - 3$ and $2x + 8 = x$ are equivalent equations.

This step is called *cancellation of equal terms.*

(iii.) *The signs of all the terms of an equation may be reversed.*

E.g., $5x - 3 = 9 - x$ becomes $-5x + 3 = -9 + x$, when both members are multiplied by -1.

8. The preceding principles apply to integral equations of any degree. In this chapter we shall confine our attention to linear equations in one unknown number.

§ 4. LINEAR EQUATIONS, IN ONE UNKNOWN NUMBER.

1. Ex. 1. Solve the equation $17x + 6 = 10x + 27$.

Transferring $10x$ to the first member and 6 to the second member, we have

$$17x - 10x = 27 - 6.$$

Uniting like terms, $7x = 21.$

Dividing by 7, $x = 3.$

Check.—Substituting 3 for x in the given equation, we obtain the identity

$$51 + 6 = 30 + 27.$$

This check is to test the accuracy of the work, and not the equivalence of the equations.

Ex. 2. Solve the equation $14 - 8x = 19 - 3x$.

Transferring terms, $-8x + 3x = 19 - 14.$

Uniting like terms, $-5x = 5.$

Dividing by -5, $x = -1.$

Ex. 3. Solve the equation $\frac{1}{2}(x + 5) - \frac{1}{3}x = \frac{1}{4}(3x - 1) + 1.$

Multiplying both members by 12, the lowest common multiple of the fractional coefficients, we obtain

$$6(x + 5) - 4x = 3(3x - 1) + 12.$$

Removing parentheses, $6x + 30 - 4x = 9x - 3 + 12.$

Transferring and uniting terms, $-7x = -21$.

Dividing by -7, $x = 3$.

Ex. 4. Solve the equation $3\frac{1}{2}(x+1) + 4\frac{1}{2}(x+1) = 16$.

Uniting terms in the first member, without clearing of fractions or removing parentheses, we have

$$8(x+1) = 16.$$

Dividing by 8, $x + 1 = 2$;

whence $x = 1$.

2. The following general directions will be found useful in preparing an equation for solution:

(i.) *Remove any fractional coefficients by multiplying both sides of the equation by the L.C.M. of their denominators.*

(ii.) *Remove any parentheses.*

(iii.) *Transfer all terms containing unknown numbers to one member of the equation, usually to the first member, and all the terms containing known numbers to the other member.*

(iv.) *Unite like terms. An equation thus prepared for solution is called the* **Normal Form** *of that equation.*

The preceding suggestions apply also to an integral equation of any degree. If the equation be linear in one unknown number, the solution is completed by dividing both members by the coefficient of the unknown number.

EXERCISES.

Solve each of the following equations:

1. $x + 2 = 3$.
2. $15 - x = -27$.
3. $17 = 9 - x$.
4. $\frac{1}{4}x = 6$.
5. $-2 = -\frac{1}{5}x$.
6. $\frac{1}{3}x = 0$.
7. $5x = 15$.
8. $11 = -22x$.
9. $4x = -16$.
10. $\frac{1}{7}x - 5 = -8$.
11. $\frac{1}{3}(x + 5) = 4$.
12. $\frac{1}{5}(x - 6) = 7$.
13. $5x + 7 = 11 + 4x$.
14. $5x - 7 = 4x + 3$.
15. $\frac{1}{2}x + 8 = -\frac{1}{2}x - 1$.
16. $7x + 8 = 4x + 15 + 2x$.
17. $-3x - 7 = -4x - 7$.
18. $15x - 8 = 20x - 8 - 4x$.
19. $36 - 9z = 116 + 11z$.
20. $61 - 5y = 7y + 85$.

21. $8x - 18 = x + 12 - 3x$.

22. $5x + 11 = 16 - 3x - 4x$.

23. $5x + 7 - 3x = 8x - 5x + 9$.

24. $-7x - 2 + 3x = -x - 4x + 3$.

25. $15x + 4 + 7x = 14x + 5 + 7x$.

26. $3x - 5 - 9x = 2x - 7 - 9x$.

27. $x - 7 = \frac{1}{5}x + \frac{1}{3}x$.

28. $\frac{1}{2}x + \frac{1}{3}x = \frac{1}{4}x - 7$.

29. $\frac{2}{3}x - \frac{5}{6}x + \frac{1}{2} = -\frac{1}{3}x$.

30. $-x + \frac{1}{2}x + \frac{1}{3}x = 11$.

31. $2x - (5x + 5) = 7$.

32. $7x - (3x - 11) = 4$.

33. $3x - 7 - (5x + 17) = 0$.

34. $3(x + 1) = -5(x - 1)$.

35. $\frac{1}{4}(x + 3) = \frac{1}{16}(3x + 16)$.

36. $\frac{1}{3}(5x - 2) - 6 = \frac{1}{5}(4x - 3)$.

37. $4x - 2(2 - x) = 6$.

38. $6x - [7x - (8x - 18)] = 16$.

39. $\frac{1}{4}(x - 2) + \frac{1}{3} - [x - \frac{1}{3}(2x - 1)] = 0$.

40. $3\frac{1}{3}[28 - (\frac{1}{8}x + 24)] = 3\frac{1}{2}(2\frac{1}{3} + \frac{1}{4}x)$.

41. $2(x + 1) - 3(x + 1) + 9(x + 1) + 18 = 7(x + 1)$.

42. $(2x + 7)(x - 3) = (x - 3)(2x + 8)$.

43. $(x + 1)(x + 2) = (x - 3)(x - 4)$.

44. $x^2 - x[1 - x - 2(3 - x)] = x + 1$.

45. $(x + 1)(x + 1) = [111 - (1 - x)]x - 80$.

46. $3 - x = 2(x - 1)(x + 2) + (x - 3)(5 - 2x)$.

47. $5(3x - 5) - 17 - 8(3x - 5) - 2(3x - 5) = 3$.

48. $-17(7x - 83) + 28(7x - 83) - 34 = 12(7x - 83)$.

49. $(16x + 5)(9x + 31) = (4x + 14)(36x + 10)$.

50. $(5x - 2)(3x - 4) = (3x + 5)(5x - 6)$.

51. $x(x + 2) + x(x + 1) = (2x - 1)(x + 3)$.

Find the remainder of each of the following divisions, and hence the value of *m* which will make the dividend exactly divisible by the divisor:

52. $(9x^2 - 3x + m) \div (x - 1)$.

53. $[4x^3 - 2x^2 + x - \frac{2}{3}(m + 1)] \div (2x + 3)$.

54. $[7x^2 - (m - 1)x + 3] \div (x + 2)$.

55. $[4x^3 - 24x^2 + (36 - m)x - 15] \div (2x - 5)$.

56. $[21x^3 - 23x^2 + (15 - m)x - 8] \div (3x - 2)$.

57. $[x^3 - 5x^2 + 3x - (m - 4)] \div (x - 5)$.

58. $[2x^3 - 5(m - 1)x^2 + 4x - 2m] \div (x - 2)$.

CHAPTER V.

PROBLEMS.

1. A **Problem** is a question proposed for solution.

Pr. **1.** The greater of two numbers is three times the less, and their sum is 84. What are the numbers?

This problem involves the *given* number 84 and two *required* numbers. The statements of the problem impose two conditions upon the values of the required numbers:

(i.) *The greater number is three times the less.*

(ii.) *The sum of the two numbers is* 84.

To solve the problem, it is necessary first to translate these relations or conditions from the verbal language of the problem into the symbolic language of Algebra, *i.e.*, to express them by means of algebraic signs and symbols.

Let x stand for the less required number.

Then, by the first condition, the greater number is,

in *verbal* language: *three times the less ;*
in *algebraic* language: $3x$.

Consequently, the required numbers are represented by x (the less) and $3x$ (the greater). The second condition is,

in *verbal* language: *the less number plus the greater is equal to* 84;
in *algebraic* language: $x + 3x = 84$.

This equation is called the *equation of the problem.*

From this equation we obtain $x = 21$, the less number.

Therefore $3x = 63$, the greater number.

Notice that this problem could have been solved by letting x stand for the greater number, and consequently $\frac{1}{3}x$ for the less. The resulting equation would then have been

$$\tfrac{1}{3}x + x = 84.$$

Whence $x = 63$, the greater number; and $\frac{1}{3}x = 21$, the less.

This method leads to an equation in which the *unknown* number is one of the *required* numbers of the problem.

Pr. 2. Find two consecutive integers whose sum is 163.

In this problem the conditions are not both *explicitly* stated. The first condition is contained in the words, *two consecutive integers.* Let x stand for the less number.

Then, by the first condition, the greater number is,

in *verbal* language: *the less number plus* 1;

in *algebraic* language: $x+1$.

The required numbers are thus represented by x (the less) and $x+1$ (the greater). The second condition is,

in *verbal* language: *the less number plus the greater is* 163;

in *algebraic* language: $x+(x+1)=163$, the equation of the problem.

From this equation we obtain $x=81$, the less number; and therefore $x+1=82$, the greater.

Pr. 3. A is 40 years old and B is 10 years old. After how many years will A be three times as old as B?

Let x stand for the required number of years, after which A will be three times as old as B.

The condition of the problem involves other unknown numbers than the required number. These we first express in terms of the required and given numbers.

In x years the *number of years in A's age* will be $40+x$; the *number of years in B's age* will be $10+x$.

The condition of the problem is,

in *verbal* language: *the number of years in A's age x years hence is equal to three times the number of years in B's age x years hence;*

in *algebraic* language: $40+x=3(10+x)$.

From this equation we obtain $x=5$, the required number. In 5 years A will be 45 years old, and B will be 15 years old.

Notice that the numbers used in the solution are *abstract* numbers; 40 is the *number of years* in A's age, not A's age.

Pr. 4. At an election at which 943 votes were cast, A and B were candidates. A received a majority of 65 votes. How many votes were cast for each candidate?

Let x stand for number of votes cast for A.

Then, by the first condition, the number of votes cast for B is,

in *verbal* language: 943 *minus the number cast for A;*

in *algebraic* language: $943 - x$.

The second condition is,

in *verbal* language: *the number of votes cast for A exceeds the number cast for B by 65;*

in *algebraic* language: $x - (943 - x) = 65$.

From this equation we obtain $x = 504$, whence $943 - x = 439$.

Pr. 5. Fifteen coins, dollars and quarter-dollars, amount to \$7.50. How many coins of each kind are there?

We take one dollar as the unit, and express parts of dollars as fractional parts of this unit.

Let x stand for the number of dollars.

Then, by the first condition, the number of quarter-dollars is,

in *verbal* language: 15 *minus the number of dollars;*

in *algebraic* language: $15 - x$.

The second condition is,

in *verbal* language: *the number of dollars plus one-fourth of the number of quarter-dollars is* $7\frac{1}{2}$;

in *algebraic* language: $x + \frac{1}{4}(15 - x) = 7\frac{1}{2}$.

From this equation we obtain $x = 5$; whence $15 - x = 10$. Evidently the total value of the coins is $5 + \frac{10}{4}$ dollars, or \$$7\frac{1}{2}$. Both conditions refer to abstract numbers; the first condition to the number of coins, the second to the number of dollars.

Pr. 6. A drove of sheep and goats, 200 animals in all, is to be sold. A offers to pay \$1.25 for each sheep and \$1.60 for each goat; B offers to pay \$1.50 for each animal. The owner of the drove accepts B's offer because he finds that it will net him \$22 more than A's offer. Find the number of sheep and goats in the drove.

Let x stand for the number of sheep.

Then, by the first condition, the number of goats is,
in *verbal* language: 200 *minus the number of sheep ;*
in *algebraic* language: $200 - x$.

The second condition involves other *unknown* numbers than the required numbers. We must express the number of dollars in A's offer and the number of dollars in B's offer in terms of the required and given numbers.

The number of dollars in A's offer is,
in *verbal* language: *the number of sheep multiplied by the number of dollars offered for each sheep, plus the number of goats multiplied by the number of dollars offered for each goat ;*
in *algebraic* language: $1.25x + 1.6(200 - x)$.

The number of dollars in B's offer is,
in *verbal* language: *the number of animals multiplied by the number of dollars offered for each animal ;*
in *algebraic* language: 200×1.5.

The second condition is,
in *verbal* language: *the number of dollars in B's offer minus the number of dollars in A's offer is* 22;
in *algebraic* language: $200 \times 1.5 - [1.25x + 1.6(200 - x)] = 22$.

From this equation we obtain $x = 120$, the number of sheep; whence $200 - x = 80$, the number of goats.

Notice again that both conditions refer to abstract numbers.

PR. 7. A box contains a certain number of pencils, of which one-third are red, one-sixth are blue, and 15 are black. How many of the pencils are red, and how many are blue?

This problem can be solved more readily by assuming for the *unknown* number of the equation another number than one of the required numbers.

Let x stand for the total number of pencils.

Then, by the first condition, the number of red pencils is $\frac{1}{3}x$, and by the second condition, the number of blue pencils is $\frac{1}{6}x$.

Finally, by the third condition,

$$\tfrac{1}{3}x + \tfrac{1}{6}x + 15 = x; \text{ whence } x = 30.$$

Therefore, $\frac{1}{3}x = 10$, the number of red pencils,
and $\frac{1}{6}x = 5$, the number of blue pencils.

Pr. 8. A number is composed of two digits whose sum is 8. If the digits be interchanged, the resulting number will exceed the original number by 18. What is the number?

In accordance with the suggestion in Pr. 7, we assume one of the digits of the required number, not the required number, as the unknown number.

Let x stand for the digit in the units' place.

Then, by the first condition, the digit in the tens' place is,

in *verbal* language: 8 *minus the digit in the units' place;*
in *algebraic* language: $8 - x$.

Therefore the original number is $10(8 - x) + x$; the second number (when the digits are interchanged) is $10x + (8 - x)$.

The second condition of the problem then is,

in *verbal* language: *the second number is equal to the original number plus* 18;
in *algebraic* language: $10x + (8 - x) = 10(8 - x) + x + 18$.

Whence, $x = 5$, the digit in the units' place;
and $8 - x = 3$, the digit in the tens' place.

The original number is $10(8 - x) + x, = 35$; the second number is $10x + 8 - x, = 53$, and $53 - 35 = 18$.

Pr. 9. A carriage, starting from a point A, travels 35 miles daily; a second carriage, starting from a point B, 84 miles behind A, travels in the same direction 49 miles daily. After how many days will the second carriage overtake the first? At what distance from B will the meeting take place?

Let x stand for the number of days after which they meet. Then the number of miles traveled by the first carriage is $35x$, and the number of miles traveled by the second carriage is $49x$.

The condition of the problem is,

in *verbal* language: *the number of miles traveled by the first carriage is equal to the number of miles traveled by the second carriage minus* 84;
in *algebraic* language: $35x = 49x - 84$.

From this equation, we obtain $x = 6$.

The distance traveled by the first carriage is 210 miles, and the distance traveled by the second carriage is 294 miles. They therefore meet 294 miles from *B*.

Pr. 10. A man asked another what time it was, and received the answer: "It is between 5 and 6 o'clock, and the minute-hand is directly over the hour-hand." What time was it?

At 5 o'clock, the minute-hand points to 12 and the hour-hand to 5. The hour-hand is therefore 25 minute-divisions in advance of the minute-hand.

Let x stand for the number of minute-divisions passed over by the minute-hand from 5 o'clock until it is directly over the hour-hand between 5 and 6 o'clock.

By the first condition, which is implied in the problem, the number of minute-divisions passed over by the hour-hand is,

in *verbal* language: *the number of minute-divisions passed over by the minute-hand minus* 25;
in *algebraic* language: $x - 25$.

The second condition, which is also implied in the problem, is,

in *verbal* language: *the number of minute-divisions passed over by the minute-hand is* 12 *times the number of minute-divisions passed over by the hour-hand;*
in *algebraic* language: $x = 12(x - 25)$.

From this equation we obtain $x = 27\frac{3}{11}$. Consequently, the two hands coincide at $27\frac{3}{11}$ minutes past 5 o'clock.

2. The beginner will find some suggestions for translating the conditions of a problem into algebraic language helpful.

(i.) *Observe what are the numbers whose values are required. It will, in general, be possible to continue the solution of the problem by representing one of these numbers by a letter, and operating upon or by that letter as if it were a known number.* See Prs. 1, 2, 3, 4, 5, 6, 9, and 10.

(ii.) *Every problem which can be solved must state, implicitly or explicitly, as many conditions as there are required numbers in the problem.*

(iii.) *The numbers involved in the statements will, in general, be not only the required numbers, but also other unknown numbers which must be expressed in terms of the required numbers.* See, in particular, Prs. 3, 6, 8.

(iv.) *Express concisely in verbal language each given condition. It is frequently necessary to modify the statement in order to adapt it to translation into algebraic language.* See Prs. 2 and 10.

(v.) *Translate each verbal statement of a condition into algebraic language. All but one of the conditions will give expressions for required numbers. The last condition will give the equation of the problem.*

(vi.) *There are problems in which other numbers than the required numbers can be used to better advantage in applying the conditions of the problems, and from which the required numbers can be readily found.* See Prs. 7, 8.

3. In applying the suggestions of Art. 2, it is important to remember that the letter x always represents an abstract number. The beginner must never put x for distance, time, weight, etc., but for the *number* of miles, of hours, of pounds, etc.

Keep in mind also that in any one equation the magnitudes of all concrete quantities of the same kind must be referred to the same unit; if x refer to a certain number of yards, then all other distances must likewise represent numbers of yards, not of miles or of feet.

EXERCISES.

1. If twice a number be added to 18, the sum will be 82. What is the number?

2. If 4 be subtracted from five times a number, the remainder will be 11. What is the number?

3. If one-fourth of a number be diminished by 5, the remainder will be 2. What is the number?

4. The sum of two consecutive even numbers is 34. What are the numbers?

5. Find the number whose double exceeds its half by 6.

6. Find three consecutive odd numbers whose sum is 57.

7. If 48 be added to a number, the sum will be equal to nine times the number. What is the number?

8. If $120 be divided between A and B so that A shall receive $20 more than B, how many dollars will each receive?

9. A sum of $2500 is divided between A and B. B receives $4 as often as A receives $1. How much does each receive?

10. Divide 75 into two parts, such that three times the first part shall be 15 greater than seven times the second.

11. Divide 190 into three parts so that the second shall be three times the first, and the third five times the second.

12. A father's age exceeds his son's by 18 years, and the sum of their ages is four times the son's age. What are their ages?

13. A man bought a horse, a carriage, and harness for $320. The horse cost five times as much as the harness, and the carriage cost twice as much as the horse. How much did each cost?

14. The deposits in a bank during three days amounted to $16,900. If the deposits each day after the first were one-third of the deposits of the preceding day, how many dollars were deposited each day?

15. A merchant, after selling one-third, one-fourth, and one-sixth of a piece of silk, has 15 yards left. How many yards were there in the piece?

16. If two trains start together and run in the same direction, one at the rate of 20 miles an hour, and the other at the rate of 30 miles an hour, after how many hours will they be 250 miles apart?

17. A teacher proposes 16 problems to a pupil. The latter is to receive 5 marks in his favor for each problem solved, and 3 marks against him for each problem not solved. If the number of marks in his favor exceed those against him by 32, how many problems will he have solved?

18. A merchant paid 30 cents a yard for a piece of cloth. He sold one-half for 35 cents a yard, one-third for 29 cents a yard, and the remainder for 32 cents a yard, gaining $18.15 by the transaction. How many yards did he buy?

19. Two men start from points 100 miles apart and travel toward each other, one at the rate of 15 miles an hour, and the other at the rate of 10 miles an hour. After how many hours will they meet, and how far will their point of meeting be from the starting point of the first?

20. A father is 32 years old, and his son is 8 years old. After how many years will the father's age be twice the son's?

21. Divide 130 into five parts so that each part shall be 12 greater than the next less part.

22. A, traveling at the rate of 20 miles a day, has four days' start of B, who travels at the rate of 25 miles a day in the same direction. After how many days will B overtake A?

23. A sum of money is equally divided among four persons. If $60 more be divided equally among six persons, the shares will be the same as before. How many dollars are divided?

24. Atmospheric air is a mixture of four parts of nitrogen with one of oxygen. How many cubic feet of oxygen are there in a room 10 yards long, 5 yards wide, and 12 feet high?

25. A merchant paid $6.15 in an equal number of dimes and five-cent pieces. How many coins of each kind did he pay?

26. A man has $4.75 in dimes and quarters, and he has 5 more quarters than dimes. How many coins of each kind has he?

27. A leaves a certain town P, traveling at the rate of 21 miles in 5 hours; B leaves the same town 3 hours later and travels in the same direction at the rate of 21 miles in 4 hours. After how many hours will B overtake A, and at what distance from P?

28. The circumference of the front and hind wheels of a wagon are 2 and 3 yards, respectively. What distance has the wagon moved when the front wheel has made 10 revolutions more than the hind wheel?

29. The sum of two numbers is 47, and their difference increased by 7 is equal to the less. What are the numbers?

30. The sum of three consecutive even numbers exceeds the least by 42. What are the numbers?

31. The sum of the two digits of a number is 4. If the digits be interchanged, the resulting number will be equal to the original one. What is the number?

32. A father is three times as old as his son, and 10 years ago he was five times as old as his son. What is the present age of each?

33. One barrel contained 48 gallons, and another 88 quarts of wine. From the first twice as much wine was drawn as from the second; the first then contained three times as much wine as the second. How much wine was drawn from each?

34. A child was born in November. On the 10th of December the number of days in its age was equal to the number of days from the 1st of November to the day of its birth, inclusive. What was the date of its birth?

35. A regiment moves from A to B, marching 20 miles a day. Two days later a second regiment leaves B for A, and marches 30 miles a day. At what distance from A do the regiments meet, A being 350 miles from B?

36. The sum of two digits of a number is 12. If the digits be interchanged, the resulting number exceeds the original one by three-fourths of the original number. What is the number?

37. Three boys, A, B, and C, have a number of marbles. A and B have 44, B and C have 43, and A and C have 39. How many marbles have they all, and how many marbles has each?

38. The tail of a fish is 4 inches long. Its head is as long as its tail and one-seventh of its body, and its body is as long as its head and one-half of its tail. How long is the fish, and how long are its head and its body?

39. A father divided his property equally among his sons. To the oldest son he gave $1000 and one-seventh of what remained; to the second son he gave $2000 and one-seventh of what was then left; to the third son he gave $3000 and one-seventh of the remainder; and so on. What was the amount of his property, and how many sons had he?

40. A man, wishing to give alms to several beggars, finds that in order to give 15 cents to each one, he must have 10 cents more than he has; but that if he were to give 12 cents to each one, he would have 14 cents left. How many beggars are there?

41. A train runs from A to B at the rate of 30 miles an hour; and returning runs from B to A at the rate of 28 miles an hour. The time required to go from A to B and return is 15 hours, including 30 minutes' stop at B. How far is A from B?

42. A cistern has 3 taps. By the first it can be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. After how many hours will the cistern be emptied, if all the taps be opened?

43. A cistern has 3 taps. By the first it can be filled in 6 hours, by the second in 8 hours, and by the third it can be emptied in 12 hours. In what time will it be filled if all the taps be opened?

44. An inlet pipe can fill a cistern in 3 hours, and an outlet pipe can empty it in 9 hours. After how many hours will the cistern be filled if both pipes be open half the time, and the outlet pipe be closed during the second half of the time?

45. In my right pocket I have as many dollars as I have cents in my left pocket. If I transfer $6.93 from my right pocket to my left, I shall have as many dollars in my left pocket as I shall have cents in my right. How much money have I in my left pocket?

46. A servant is to receive $170 and a dress for one year's services. At the end of 7 months she leaves her place and receives $95 and the dress. What is the value of the dress?

47. A farmer found that his supply of feed for his cows would last only 14 weeks. He therefore sold 60 cows, and his supply then lasted 20 weeks. How many cows had he?

48. At 6 o'clock the hands of a clock are in a straight line. At what time between 7 and 8 o'clock will they be again in a straight line? At what time between 9 and 10 o'clock?

49. A cistern has 3 pipes which can empty it in 6, 8, and 10 hours respectively. After all three pipes have been open for 2 hours they have discharged 94 gallons. What is the capacity of the cistern?

50. At what time between 10 and 11 o'clock are the minute-hand and the hour-hand of a clock at right angles to each other? Find two solutions. At what time between 12 and 1 o'clock?

51. At what time between 3 and 4 o'clock will the minute-hand of a clock be 5 minute-divisions in advance of the hour-hand? At what time 17 minute-divisions?

A watch has the second-hand attached at the same point as the hour- and the minute-hand:

52. At what time between 1 and 2 o'clock is the second-hand over the minute-hand? At what time between 8 and 9 o'clock?

53. At what time between 11 and 12 o'clock does the second-hand of a watch bisect the angle between the hour- and the minute-hand? At what time between 4 and 5 o'clock?

54. A woman sells ½ an apple more than one-half of her apples. She next sells ½ an apple more than one-half of the apples not yet sold, and then has 6 apples left. How many apples had she at first?

55. A steamer and a sailing vessel are both to sail from M to N. The steamer sails 40 miles every 3 hours, and the sailing vessel 24 miles in the same time. The sailing vessel has traveled 13½ miles when the steamer sails, and arrives at N 5 hours later than the steamer. How long is the steamer in sailing from M to N, and how far is M from N?

56. A wall can be built by 20 workmen in 11 days, or by 30 other workmen in 7 days. If 22 of the first class work together with 21 of the second class, after how many days will the work be completed?

57. In a certain family each son has as many brothers as sisters, but each daughter has twice as many brothers as sisters. How many children are in the family?

58. A merchant's investment yields him yearly 33⅓% profit. At the end of each year, after deducting $1000 for personal expenses, he adds the balance of his profits to his invested capital. At the end of three years his capital is twice his original investment. How much did he invest?

59. I have in mind a number of six digits, the last one on the left being 1. If I bring this digit to the first place on the right, I shall obtain a number which is three times the number I have in mind. What is the number?

60. A dog caught sight of a hare at a distance of 50 dog's leaps. The dog makes 3 leaps while the hare makes 4 leaps, but the length of two dog's leaps is equal to the length of 3 hare's leaps. How many leaps will the hare make before the dog overtakes him?

CHAPTER VI.

TYPE-FORMS.

We shall in this chapter consider a number of products and quotients which are of frequent occurrence. They are called **Type-Forms.**

§ 1. TYPE-FORMS IN MULTIPLICATION.

The Square of an Algebraic Expression.

1. By actual multiplication, we have

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2.$$

That is, *the square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two numbers, plus the square of the second number.*

E.g., $(2x+5y)^2 = (2x)^2 + 2(2x)(5y) + (5y)^2$
$= 4x^2 + 20xy + 25y^2.$

2. By actual multiplication, we have

$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2.$$

That is, *the square of the difference of two numbers is equal to the square of the first number, minus twice the product of the two numbers, plus the square of the second number.*

E.g., $(3x-7y)^2 = (3x)^2 - 2(3x)(7y) + (7y)^2$
$= 9x^2 - 42xy + 49y^2.$

Observe that this type-form is equivalent to that of Art. 1, since $a - b = a + (-b)$.

E.g., $(3x-7y)^2 = (3x)^2 + 2(3x)(-7y) + (-7y)^2$
$= 9x^2 - 42xy + 49y^2$, as above.

The signs of all the terms of an expression which is to be squared may be changed without changing the result.

For, $(a-b)^2 = [-(b-a)]^2 = (b-a)^2.$

3. We have, by Art. 1,

$$(a+b+c)^2=[(a+b)+c]^2=(a+b)^2+2(a+b)c+c^2$$
$$=a^2+2\,ab+b^2+2\,ac+2\,bc+c^2.$$

Therefore $\boldsymbol{(a+b+c)^2=a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.}$

In like manner

$$\boldsymbol{(a+b-c)^2=a^2+b^2+c^2+2\,ab-2\,ac-2\,bc.}$$
$$\boldsymbol{(a-b-c)^2=a^2+b^2+c^2-2\,ab-2\,ac+2\,bc.}$$

By repeated application of this principle we can obtain the square of a multinomial of any number of terms. We have

$$(a+b+c+d)^2=[(a+b+c)^2+d]^2$$
$$=a^2+b^2+c^2+2\,ab+2\,ac+2\,bc+2(a+b+c)d+d^2$$
$$=a^2+b^2+c^2+d^2+2\,ab+2\,ac+2\,ad+2\,bc+2\,bd+2\,cd.$$

That is, *the square of a multinomial is equal to the sum of the squares of the terms, plus the algebraic sum of twice the product of each term by each term which follows it.*

E.g., $(3\,x+5\,y-7\,z)^2=(3\,x)^2+(5\,y)^2+(-7\,z)^2+2(3\,x)(5\,y)$
$$+2(3\,x)(-7\,z)+2(5\,y)(-7\,z)$$
$$=9\,x^2+25\,y^2+49\,z^2+30\,xy-42\,xz-70\,yz.$$

EXERCISES I.

Write, without performing the actual multiplications, the values of

1. $(a+1)^2$. **2.** $(2-b)^2$. **3.** $(x+5)^2$.

4. $(2\,a+3\,b)^2$. **5.** $(5\,x-7\,y)^2$. **6.** $(3\,ax+2\,by)^2$.

7. $(a^2+\frac{1}{3}\,ab)^2$. **8.** $(\frac{1}{2}\,x^2+\frac{2}{3}\,y)^2$. **9.** $(2\,xy-4)^2$.

10. $(2\,m^2-3\,n^2)^2$. **11.** $(5\,x^2y^2-3\,z^3)^2$. **12.** $(5\,x^{n+1}+7\,x)^2$.

13. $(6\,a^nb^m-5\,a^mb^n)^2$. **14.** $(2\,a^{m+n}-3\,a^{m-n})^2$. **15.** $[a+b(x-1)]^2$.

16. $[2(x+1)-3(y+z)]^2$. **17.** $(2\,x-3\,y+7)^2$. **18.** $(m^4+n^2-1)^2$.

19. $(a^3+a^2+1)^2$. **20.** $(2\,ab+3\,a^2+4\,b^2)^2$.

21. $(x^2-3\,xy+y^2)^2$. **22.** $(xy-xz+yz-3)^2$.

Simplify the following expressions:

23. $a^2+b^2-(a-b)^2$.

24. $x^2+y^2-4\,x+6\,y+3$, when $x=a+1$, $y=a-2$.

25. $(a+b+c)^2+(a-b-c)^2+(a-b+c)^2+(a+b-c)^2$.

26. $(a+b-c)(a+b)+(a-b+c)(a+c)+(b+c-a)(b+c)$.

Verify the following identities:

27. $(a^2+b^2)(x^2+y^2)-(ax+by)^2=(ay-bx)^2$.

28. $(a+b+c)^2+(a-b)^2+(a-c)^2+(b-c)^2=3(a^2+b^2+c^2)$.

29. $a^2+b^2+4c^2+2ab+8bc=4(a+c)^2$, when $b=a$.

30. $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2$
$+(a+b-c-d)^2=4(a^2+b^2+c^2+d^2)$.

31. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-a)(s-c)$
$+c(s-a)(s-b)=abc$, when $2s=a+b+c$.

Product of the Sum and Difference of Two Numbers.

4. By actual multiplication, we have

$$(a+b)(a-b)=a^2-b^2.$$

That is, *the product of the sum of two numbers and the difference of the same numbers, taken in the same order, is equal to the square of the first, minus the square of the second.*

Ex. 1. $(2x+3y)(2x-3y)=(2x)^2-(3y)^2=4x^2-9y^2$.

The product of two multinomials can frequently be brought under this type-form by properly grouping terms.

Ex. 2.
$$\begin{aligned}(x^2+x+1)(x^2-x+1)&=[(x^2+1)+x][(x^2+1)-x]\\&=(x^2+1)^2-x^2\\&=x^4+2x^2+1-x^2\\&=x^4+x^2+1.\end{aligned}$$

Ex. 3.
$$\begin{aligned}(x-y+z)(x+y-z)&=[x-(y-z)][x+(y-z)]\\&=x^2-(y-z)^2\\&=x^2-(y^2-2yz+z^2)\\&=x^2-y^2-z^2+2yz.\end{aligned}$$

EXERCISES II.

Find, without performing the actual multiplications, the values of

1. $(x+2)(x-2)$.
2. $(2a-3)(2a+3)$.
3. $(a-\frac{2}{3})(a+\frac{2}{3})$.
4. $(5x+4y)(5x-4y)$.
5. $(3a^2+\frac{1}{2}ab)(3a^2-\frac{1}{2}ab)$.
6. $(-3x^2+7)(3x^2+7)$.
7. $(2ax^2-3a^2x)(2ax^2+3a^2x)$.
8. $(3a^n+7b^m)(3a^n-7b^m)$.
9. $(5mn^2+2m^2n)(-5mn^2+2m^2n)$.
10. $(-5x^{n+1}+9x^{n-1})(5x^{n+1}+9x^{n-1})$.
11. $[a^2+6(a+b)][a^2-6(a+b)]$.

12. $(x+y+5)(x+y-5)$. **13.** $(4a-3b-7)(4a-3b+7)$.
14. $(x^2+y^2+z^2)(-x^2+y^2+z^2)$. **15.** $(a^2-ab+b^2)(a^2+ab+b^2)$.
16. $(x^2+2x-1)(x^2-2x-1)$. **17.** $(x^4-x^2+1)(x^4+x^2-1)$.
18. $(-a^2-b^2+3)(a^2-b^2+3)$. **19.** $(a^2-b^2-c^2)(a^2+b^2+c^2)$.
20. $(1+2a+3b+4c)(1+2a-3b-4c)$.
21. $(a+b+c-d)(a+b-c+d)$.
22. $(a^3-3a^2x+3ax^2-x^3)(a^3+3a^2x+3ax^2+x^3)$.
23. $(x^2-y^2-z^2-w^2)(x^2+y^2+z^2-w^2)$.
24. $(-x^3+x^2-2x-1)(x^3+x^2+2x-1)$.

Simplify the following expressions:

25. $(1+x)^2-(1-x)(1+x)$. **26.** $(2x+3y)^2(2x-3y)^2$.
27. $(1-ab)^2(1+ab)^2$. **28.** $(x-3)(x-1)(x+1)(x+3)$.
29. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)$.
30. $(x^2-1)(x^8+1)(x^4+1)(x^2+1)$.
31. $(a^2+2ab)(a^2-2ab)(a^8+16a^4b^4)(a^4+4a^2b^2)$.
32. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.
33. $(a+b-c)(a+c-b)(b+c-a)(a+b+c)$.
34. $(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)$.

The Product $(x+a)(x+b)$.

5. By actual multiplication, we have

$$(x+a)(x+b)=x^2+(a+b)x+ab;$$
$$(x+a)(x-b)=x^2+(a-b)x-ab;$$
$$(x-a)(x-b)=x^2-(a+b)x+ab.$$

That is, *the product of two binomials, having the same first term, is equal to the square of this term, plus the product of the algebraic sum of the second terms by the common first term, plus the product of the second terms.*

Ex. 1. $(x+3)(x+5)=x^2+8x+15$.

Ex. 2. $(ax-b)(ax-c)=a^2x^2-(b+c)ax+bc$.

Ex. 3. $$\begin{aligned}(a+b+5)(a+b-3)&=[(a+b)+5][(a+b)-3]\\&=(a+b)^2+2(a+b)-15.\end{aligned}$$

EXERCISES III.

Find, without performing the actual multiplications, the values of the following indicated products:

1. $(x+7)(x+4)$. **2.** $(x-5)(x-4)$. **3.** $(a-6)(a+8)$.
4. $(x-3)(x+2)$. **5.** $(5+b)(7+b)$. **6.** $(3-x)(4-x)$.
7. $(6+y)(y-3)$. **8.** $(2x+1)(2x+3)$.
9. $(ay+1)(ay+5)$. **10.** $(3xy-8)(3xy-7)$.
11. $(5a-3b)(5a-7b)$. **12.** $(2xy+3z)(2xy-4z)$.
13. $(x^2-7)(x^2-5)$. **14.** $(2x^3-11)(2x^3+4)$.
15. $(x^{m+1}-4)(x^{m+1}-7)$. **16.** $(x+y-3)(x+y-5)$.
17. $(ax+by-c)(ax+by-2c)$. **18.** $(x^2-2x+7)(x^2-2x-3)$.

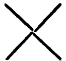
The Product $(ax+b)(cx+d)$.

6. By actual multiplication, we obtain

$$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd.$$

In this type-form that part of the multiplication which gives the middle term of the type-form may be represented concisely by the following arrangement:

$$cx+d$$



$$\frac{ax+b}{(ad+bc)x}$$

The products of the terms connected by the cross lines are called *cross-products*, and their sum is the middle term of the given trinomial.

That is, *the product of two binomials, arranged to powers of a common letter, is equal to the product of the first terms, plus the sum of the cross-products, plus the product of the last terms.*

Ex. 1. $(7x-5y)(2x+3y)=7x\cdot 2x+(7\cdot 3-5\cdot 2)xy-5y\cdot 3y$
$=14x^2+11xy-15y^2$.

EXERCISES IV.

Find, without performing the actual multiplications, the values of the following indicated products:

1. $(3a+1)(5a+2)$. **2.** $(7x-3)(3x-1)$. **3.** $(5x+7)(3x-2)$.
4. $(2x-9)(5x+1)$. **5.** $(2x+15)(4x-5)$. **6.** $(11a-3)(9a+7)$.

7. $(2a+b)(3a-b)$.
8. $(2a-b)(3a+b)$.
9. $(3x-y)(2x-y)$.
10. $(7a+3b)(5a-2b)$.
11. $(6x-7y)(3x+2y)$.
12. $(5x-3z)(2x+5z)$.
13. $(7y+2u)(8y-7u)$.
14. $(2ab-x)(3ab+x)$.
15. $(5mn-3p)(6mn+7p)$.
16. $(9m^2-3)(8m^2+11)$.
17. $(3x^2+5y^2)(2x^2-3y^2)$.
18. $[3(a+b)+5][5(a+b)-2]$.
19. $[2(x-y)-7][3(x-y)+2]$.

The Cube of an Algebraic Expression.

7. By actual multiplication, we have

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (1)$$

and

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3. \qquad (2)$$

That is, *the cube of the sum of two numbers is equal to the cube of the first number, plus three times the product of the square of the first number by the second, plus three times the product of the first number by the square of the second, plus the cube of the second.*

A similar statement can be made for (2).

Ex. 1. $(5x+3y)^3 = (5x)^3 + 3(5x)^2(3y) + 3(5x)(3y)^2 + (3y)^3$
$$= 125x^3 + 225x^2y + 135xy^2 + 27y^3.$$

Ex. 2. $(x+2y-3z)^3 = [(x+2y)-(3z)]^3$
$$\begin{aligned} &= (x+2y)^3 - 3(x+2y)^2(3z) \\ &\quad + 3(x+2y)(3z)^2 - (3z)^3 \\ &= x^3 + 6x^2y + 12xy^2 + 8y^3 - 9x^2z - 36xyz \\ &\quad - 36y^2z + 27xz^2 + 54yz^2 - 27z^3. \end{aligned}$$

Observe that (2) is similar to (1), since

$$(a-b)^3 = [a+(-b)]^3.$$

Also that

$$(a-b)^3 \doteq -(b-a)^3.$$

EXERCISES V.

Write, without performing the actual multiplications, the values of

1. $(a+1)^3$.
2. $(a-2)^3$.
3. $(2x+3)^3$.
4. $(3-4y)^3$.
5. $(a+2b)^3$.
6. $(3a-b)^3$.
7. $(ax+by)^3$.
8. $(2x-3yz)^3$.
9. $(x^2+3x)^3$.

10. $(a^2n^2 - 6an)^3$.
11. $(2xm^3 + 5x^2m^2)^3$.
12. $(\frac{5}{3}a^2x - \frac{1}{5}ax^2)^3$.
13. $(7x^n + 2x^{n-1})^3$.
14. $(a^nb - 5ab^{n+1})^3$.
15. $(a^2 + a + 1)^3$.
16. $(3 - x - x^2)^3$.
17. $(\frac{5}{3}x^2 - x + \frac{1}{2})^3$.
18. What is the value of $x^3 - 3x^2 + 3x - 1$, when $x = m + 1$?
19. What is the value of $x^3 - 3x^2 + 3x - 1$, when $x = y - 2$?

Verify the following identities:

20. $(a + b + c)^3 - 3(a + b)(b + c)(c + a) = a^3 + b^3 + c^3$.
21. $(a - b)^3 + 3(a - b)^2(a + b) + (a + b)^3 + 3(a - b)(a + b)^2 = 8a^3$.
22. $(a + b + c)^3 - (b + c - a)^3 - (c + a - b)^3 - (a + b - c)^3 = 24abc$.

Higher Powers of a Binomial.

8. By actual multiplication, we have

$$\begin{aligned}
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4;\\
(a-b)^4 &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4;\\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5;\\
(a-b)^5 &= a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5;\\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6;\\
(a-b)^6 &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.
\end{aligned}$$

The result of performing the indicated operation in a power of a binomial is called the **Expansion** of that power of the binomial.

In the preceding expansions the following laws are evident:

(i.) *The number of terms exceeds the binomial exponent by* 1.

(ii.) *The exponent of* a *in the first term is equal to the binomial exponent, and decreases by* 1 *from term to term.*

(iii.) *The exponent of* b *in the second term is* 1 *and increases by* 1 *from term to term, and in the last term is equal to the binomial exponent.*

(iv.) *The coefficient of the first term is* 1, *and that of the second term, except for sign, is equal to the binomial exponent.*

(v.) *The coefficient of any term after the second is obtained, except for sign, by multiplying the coefficient of the preceding term by the exponent of* a *in that term, and dividing the product by a number greater by* 1 *than the exponent of* b *in that term.*

E.g., the coefficient of the *fifth* term in the expansion of

$$(a+b)^5 \text{ is } 10 \times 2 \div 4 = 5.$$

(vi.) *The signs of the terms are all positive when the terms of the binomial are both positive; the signs of the terms alternate, + and −, when one of the terms of the binomial is negative.*

Observe, as a check:

(vii.) *The sum of the exponents of* a *and* b *in any term is equal to the binomial exponent.*

(viii.) *The coefficients of two terms equally distant from the beginning and the end of the expansion are equal.*

In a subsequent chapter the above laws will be proved to hold for any positive integral power of the binomial.

Ex. 1.
$$\begin{aligned}(2x-3y)^4 &= (2x)^4 - 4(2x)^3(3y) + 6(2x)^2(3y)^2 \\ &\quad - 4(2x)(3y)^3 + (3y)^4 \\ &= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.\end{aligned}$$

EXERCISES VI.

Find the values of the following powers:

1. $(1+x)^4$. 2. $(3-4a)^4$. 3. $(2x+3y)^5$.
4. $(3m-2n)^5$. 5. $(2a+1)^6$. 6. $(3x-2y)^6$.
7. $(a^2+ab)^4$. 8. $(x^3-x)^5$. 9. $(2m^3n-mn^3)^6$.
10. $(a^{n+1}-b)^4$. 11. $(a^{n-1}b+ab^n)^5$. 12. $(a^{m+1}x^{n-1}-a^{n-1}x^{m+1})^6$.
13. Verify the identity

$$[(a-b)^2+(b-c)^2+(c-a)^2]^2 = 2[(a-b)^4+(b-c)^4+(c-a)^4].$$

§ 2. TYPE-FORMS IN DIVISION.

Quotient of the Sum or the Difference of Like Powers of Two Numbers by the Sum or the Difference of the Numbers.

1. By actual division, we have

$$(a^2-b^2) \div (a+b) = a-b \text{ and } (a^2-b^2) \div (a-b) = a+b.$$

That is, *the difference of the squares of two numbers is exactly divisible by the sum of the numbers, and also by the difference of the numbers, taken in the same order; the quotient in the first case is the difference of the two numbers, taken in the same order, and in the second case is the sum of the two numbers.*

Ex. 1. $(9-25x^2)\div(3+5x)=3-5x.$

Ex. 2. $(16x^4-81y^6)\div(4x^2-9y^3)=4x^2+9y^3.$

Ex. 3. $$\begin{aligned}&(x^2+y^2-z^2-2xy)\div(x-y-z)\\&=(x^2-2xy+y^2-z^2)\div(x-y-z)\\&=[(x-y)^2-z^2]\div[(x-y)-z]\\&=x-y+z.\end{aligned}$$

2. By actual division, we have

$$\text{(i.)}\ \begin{cases}(a^3+b^3)\div(a+b)=a^2-ab+b^2;\\(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4.\end{cases}$$

$$\text{(ii.)}\ \begin{cases}(a^3-b^3)\div(a-b)=a^2+ab+b^2;\\(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4.\end{cases}$$

$$\text{(iii.)}\ \begin{cases}(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3;\\(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3.\end{cases}$$

The above identities, and the identities in Art. 1, illustrate the following principles, which will be proved in Art. 6:

(i.) *The sum of the like odd powers of two numbers is exactly divisible by the sum of the numbers.*

(ii.) *The difference of the like odd powers of two numbers is exactly divisible by the difference of the numbers, taken in the same order.*

(iii.) *The difference of the like even powers of two numbers is exactly divisible by the sum, and also by the difference of the numbers, taken in the same order.*

(iv.) *When the divisor is a sum, the signs of the terms of the quotient alternate,* + *and* −.

(v.) *When the divisor is a difference, the signs of the terms of the quotient are all* +.

(vi.) *In the first term of the quotient the exponent of* a *is less by* 1 *than its exponent in the dividend, and decreases by* 1 *from term to term.*

(vii.) *The exponent of* b *is* 1 *in the second term of the quotient, and increases by* 1 *from term to term.*

Observe that the quotient is homogeneous in a and b, of degree less by 1 than the degree of the dividend.

Ex. 1. $(8\,x^3+\frac{1}{125})\div(2\,x+\frac{1}{5})=(2\,x)^2-(2\,x)(\frac{1}{5})+(\frac{1}{5})^2$
$=4\,x^2-\frac{2}{5}\,x+\frac{1}{25}.$

Ex. 2. $(32\,y^{10}-x^{15})\div(2\,y^2-x^3)$
$=(2\,y^2)^4+(2\,y^2)^3(x^3)+(2\,y^2)^2(x^3)^2+(2\,y^2)(x^3)^3+(x^3)^4$
$=16\,y^8+8\,y^6x^3+4\,y^4x^6+2\,y^2x^9+x^{12}.$

Notice that in the type-forms each term, *beginning with the second*, is equal to the preceding term $\times\, b\div a$ when the divisor is $a-b$, and $\times\,(-b)\div a$ when the divisor is $a+b$. Thus, in Ex. 2:

$$16\,y^8=32\,y^{10}\div 2\,y^2;\ \ 8\,y^6x^3=16\,y^8\times x^3\div 2\,y^2;$$
$$4\,y^4x^6=8\,y^6x^3\times x^3\div 2\,y^2;\ \ 2\,y^2x^9=4\,y^4x^6\times x^3\div 2\,y^2;$$
$$x^{12}=2\,y^2x^9\times x^3\div 2\,y^2.$$

It is important to notice that, as we shall prove in Art. 6, *the sum of the like even powers of two numbers is not exactly divisible by either the sum or the difference of the numbers.*

E.g., a^4+b^4 is not divisible either by $a+b$ or by $a-b$.

EXERCISES VII.

Find the values of the following quotients, without performing the actual divisions:

1. $(x^2-1)\div(x-1)$. **2.** $(25-x^2)\div(5+x)$.
3. $(4\,a^2-9)\div(2\,a-3)$. **4.** $(\frac{1}{9}-x^2y^2)\div(\frac{1}{3}+xy)$.
5. $(16\,x^2-9\,y^2)\div(4\,x-3\,y)$. **6.** $(64\,a^2b^2-121\,c^2)\div(8\,ab+11\,c)$.
7. $(x^4-1)\div(x^2+1)$. **8.** $(4\,a^4-b^2)\div(2\,a^2-b)$.
9. $(x^{2n}-1)\div(x^n-1)$. **10.** $(a^{4n}-16\,b^{16})\div(a^{2n}+4\,b^8)$.
11. $(x^{2n+2}-4)\div(x^{n+1}+2)$. **12.** $(a^{8n}-b^{4n+4})\div(a^{4n}-b^{2n+2})$.
13. $[(a+b)^2-1]\div(a+b+1)$. **14.** $[4-(a+b)^2]\div(2-a-b)$.
15. $(a^2-2\,ab+b^2-1)\div(a-b+1)$. **16.** $(a^2-n^2-p^2+2\,np)\div(a-n+p)$.
17. $(p^2-r^2-4-4\,r)\div(p-r-2)$.
18. $[(a^2+2\,ab+b^2)x^6-y^4]\div[(a+b)x^3+y^2]$.
19. $(x^4+2\,x^2y^2+y^4-z^2-2\,zu-u^2)\div(x^2+y^2+u+z)$.
20. $(a^2-b^2+2\,bz-2\,ax+x^2-z^2)\div(a-x-b+z)$.
21. $(x^3y^3+1)\div(xy+1)$. **22.** $(1-a^3)\div(1-a)$.
23. $(x^3+125)\div(x+5)$. **24.** $(8\,a^3-27)\div(3-2\,a)$.
25. $(a^6-1)\div(a^2-1)$. **26.** $(a^{12}+27)\div(a^4+3)$.
27. $(8\,m^{15}n^3-p^{12})\div(2\,m^5n-p^4)$. **28.** $(a^{3n}-1)\div(a^n-1)$.
29. $(343\,x^{3m-3}-y^{6n})\div(7\,x^{m-1}-y^{2n})$. **30.** $[(x+y)^3-z^6]\div(x+y-z^2)$.

31. $(x^3 + 8y^3 + z^3 + 6x^2y + 12xy^2) \div (x + 2y + z)$.

32. $(x^4 - 1) \div (x - 1)$.

33. $(1 - 16a^4) \div (1 + 2a)$.

34. $(\frac{1}{81}x^4 - 16y^4) \div (\frac{1}{3}x - 2y)$.

35. $(81x^8 - 16y^8) \div (3x^2 + 2y^2)$.

36. $(x^8y^{12} - 256z^{16}) \div (x^2y^3 - 4z^4)$.

37. $(x^{4n} - y^{4n}) \div (x^n + y^n)$.

38. $(a^5 + 1) \div (a + 1)$.

39. $(32x^5 + y^5) \div (2x + y)$.

40. $(1 - x^5) \div (x - 1)$.

41. $(243a^5b^5 - c^5) \div (3ab - c)$.

42. $(x^{10}y^5 + 32z^{15}) \div (x^2y + 2z^3)$.

43. $(a^{10}b^{15}c^{20} - d^{25}) \div (a^2b^3c^4 - d^5)$.

44. $(x^{5n} - 1) \div (x^n - 1)$.

45. $(1 + a^{5m}) \div (1 + a^m)$.

46. $(a^6 - b^6) \div (a - b)$.

47. $(a^7 + b^7) \div (a + b)$.

48. $(a^{11} + 1) \div (a + 1)$.

49. $(a^{12} - 1) \div (a - 1)$.

50. $(a^{10} - \frac{1}{32}b^5) \div (a^2 - \frac{1}{2}b)$.

51. $(a^7 - x^{14}) \div (a - x^2)$.

52. $(x^{30} - 1) \div (x^5 + 1)$.

53. $(a^{14}x^{7n} + b^{14m}) \div (a^2x^n + b^{2m})$.

Of what divisions are the following expressions the quotients:

54. $x^2 + x + 1$.

55. $a^2 - ab + b^2$.

56. $x^3 - x^2 + x - 1$.

57. $a^4 + a^3 + a^2 + a + 1$.

58. $x^3y^3 + mx^2y^2 + m^2xy + m^3$.

59. $x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1$.

The Remainder Theorem.

3. If $3x^3 - 4x^2 - 6x + 7$ be divided by $x - 2$, we obtain a partial quotient $3x^2 + 2x - 2$ and a remainder 3.

If now 2 be substituted for x in the given expression, we obtain

$$3 \times 2^3 - 4 \times 2^2 - 6 \times 2 + 7 = 3, \text{ the above remainder.}$$

This example illustrates the following principle:

If an expression, arranged to powers of a letter of arrangement, say x, be not exactly divisible by $x - a$, the remainder of the division is equal to the result of substituting a for x in the given expression.

Let the given expression be of the form

$$Ax^n + Bx^{n-1} + \cdots + Ux + V,$$

in which n is a positive integer.

Let Q stand for the partial quotient of the division by $x - a$, and R for the remainder. Then, by Ch. III., § 4, Art. 13, we have

$$Ax^n + Bx^{n-1} + \cdots + Ux + V = Q(x - a) + R.$$

If now a be substituted for x in the last equation, we obtain

$$\begin{aligned} Aa^n + Ba^{n-1} + \cdots + Ua + V &= Q(a - a) + R \\ &= Q \cdot 0 + R = R. \end{aligned}$$

That is, the remainder, R, of dividing the given expression by $x - a$, is equal to the result of substituting a for x in the expression.

4. From the principle of the preceding article we derive the following: *If an expression, arranged to powers of a letter of arrangement, say x, be exactly divisible by $x - a$, the result of substituting a for x in the given expression is* 0; *and conversely.*

For if the division be exact, the remainder is 0, and therefore the result of the substitution is 0.

E.g., $3x^3 - 4x^2 - 6x + 4$ is exactly divisible by $x - 2$.

Substituting 2 for x, we obtain $3 \times 2^3 - 4 \times 2^2 - 6 \times 2 + 4 = 0$.

5. *If an expression be arranged to descending powers of a letter of arrangement, the following is a convenient method of substituting a particular value for the letter of arrangement.*

Ex. 1. Substitute 2 for x in $3x^3 - 4x^2 - 6x + 7$.

We have $3x^3 = 3x \cdot x^2 = 6x^2$, when $x = 2$;

therefore $3x^3 - 4x^2 = 6x^2 - 4x^2 = 2x^2$, when $x = 2$;

then $2x^2 = 2x \cdot x = 4x$, when $x = 2$;

and $4x - 6x = -2x = -4$, when $x = 2$;

finally $-4 + 7 = 3$, the result of the substitution.

Ex. 2. Is $x^3 + x^2 - x + 7$ exactly divisible by $x + 2$?

Since $x + 2 = x - (-2)$, we substitute -2 for x. We then have

$x^3 = -2x^2$; $-2x^2 + x^2 = -x^2 = 2x$; $2x - x = x = -2$; $-2 + 7 = 5$.

Therefore $x^3 + x^2 - x + 7$ is not exactly divisible by $x + 2$, and the remainder of the division is 5.

EXERCISES VIII.

Prove that the following dividends are exactly divisible by the corresponding divisors, without performing the divisions:

1. $(x^2 - 3x + 2) \div (x - 2)$.
2. $(x^3 - 3x + 2) \div (x - 1)$.
3. $(x^3 - 18x - 35) \div (x - 5)$.
4. $(x^3 + 2x^2 - x - 2) \div (x + 1)$.
5. $(x^3 + 21x + 342) \div (x + 6)$.
6. $(2x^3 + 3x^2 + 3x + 1) \div (x + \frac{1}{2})$.
7. $(x^6 - 6x^4 - 19x^2 + 84) \div (x^2 - 3)(x^2 + 4)$.
8. $(x^6 + 4x^4 + x^2 - 6) \div (x^2 - 1)(x^2 + 2)$.
9. $(2x^4 + 4ax^3 - 5a^2x^2 - 3a^3x + 2a^4) \div (x - a)$.
10. $(x^4 + 3a^2x^2 + 5a^3x + a^4) \div (x + a)$.

Find the remainders of the following indicated divisions, without performing the divisions:

11. $(2x^3 - 7x^2 + 6x - 15) \div (x + 4)$.
12. $(5x^4 - 11x^2 + 2x - 7) \div (x - 2)$.
13. $(17x^3 - 2x^2 + 4x - 3) \div (x - \frac{2}{5})$.

14. Prove that $(x+1)^m+(x-1)^m$ is exactly divisible by x when m is odd.

15. Prove that $(a+b+c)^3-(a^3+b^3+c^3)$ is exactly divisible by $(a+b)(a+c)(b+c)$.

16. Prove that $x^qy^r+y^qz^r+z^qx^r-x^ry^q-y^rz^q-z^rx^q$ is exactly divisible by $(x-y)(x-z)(y-z)$.

6. We are now prepared to prove the following principles enunciated in Art. 2:

(i.) *a^n+b^n is exactly divisible by $a+b$, but not by $a-b$, when n is odd.*

The quotient is

$$a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots+a^2b^{n-3}-ab^{n-2}+b^{n-1}.$$

(ii.) *a^n-b^n is exactly divisible by $a-b$, but not by $a+b$, when n is odd.*

The quotient is

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+a^2b^{n-3}+ab^{n-2}+b^{n-1}.$$

(iii.) *a^n-b^n is exactly divisible by $a+b$, and by $a-b$, when n is even.*

The quotient is, when $a+b$ is the divisor,

$$a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots-a^2b^{n-3}+ab^{n-2}-b^{n-1};$$

and when $a-b$ is the divisor,

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+a^2b^{n-3}+ab^{n-2}+b^{n-1}.$$

(iv.) *a^n+b^n is not exactly divisible by either $a+b$ or $a-b$, when n is even.*

For if $-b$ be substituted for a in a^n+b^n, we obtain

$$(-b)^n+b^n=0, \text{ only when } n \textit{ is odd.}$$

Therefore, a^n+b^n is exactly divisible by $a+b$, only when n is odd.

If b be substituted for a in a^n+b^n, we obtain $b^n+b^n, \neq 0$.

Therefore, a^n+b^n is not divisible by $a-b$.

In like manner the other principles can be proved.

It is evident that the forms of the quotients considered in this article, obtained by actual division, conform to principles (iv.)–(vii.), Art. 2.

CHAPTER VII.

PARENTHESES.

1. The use of parentheses has been briefly discussed in Ch. II., § 2, Arts. 8–9. It is frequently necessary to employ more than two sets of parentheses in the same chain of operations, and to distinguish them the following forms are used:

Parentheses, ( ); **Brackets,** [ ]; **Braces,** { }.

A **Vinculum** is a line drawn over an expression, and is equivalent to parentheses inclosing it.

E.g., $(a+b)(c-d)=\overline{a+b}\cdot\overline{c-d}$.

If more forms of parentheses than the above are needed, one or more of them is made larger and heavier.

Removal of Parentheses.

2. The principles given in Ch. II., § 2, Art. 8, are to be applied successively when several sets of parentheses are to be removed from a given expression. In removing parentheses we may begin either with the inmost or with the outmost.

3. The following examples will illustrate the method of removing parentheses, beginning with the inmost:

Ex. **1.**

$$\begin{aligned}&4a-\{3a+[2a-(a-1)]\}\\&=4a-\{3a+[2a-a+1]\}\\&=4a-\{3a+a+1\}\\&=4a-4a-1=-1.\end{aligned}$$

Ex. **2.**

$$\begin{aligned}&[b^2-\{(a^2+b)a-(a^2-b)b-a^2(a-b)\}]^3\\&=[b^2-\{a^3+ab-a^2b+b^2-a^3+a^2b\}]^3\\&=[b^2-\{ab+b^2\}]^3\\&=[b^2-ab-b^2]^3=(-ab)^3=-a^3b^3.\end{aligned}$$

4. The method of removing parentheses, beginning with the outmost, may be illustrated by the following example.

Notice that the part which is free from parentheses is simplified at the same time that the next parentheses are removed.

$$x^2 - \{2x^2 - [3x^2 + (4x^2 - \overline{5x^2 - 1})]\}$$
$$= x^2 - 2x^2 + [3x^2 + (4x^2 - \overline{5x^2 - 1})], \text{ removing braces.}$$
$$= - x^2 + 3x^2 + (4x^2 - \overline{5x^2 - 1}), \text{ removing brackets,}$$
$$= 2x^2 + 4x^2 - \overline{5x^2 - 1}, \text{ removing parentheses,}$$
$$= 6x^2 - 5x^2 + 1, \text{ removing vinculum,}$$
$$= x^2 + 1.$$

EXERCISES I.

Simplify the following expressions by removing parentheses:

1. $a + 2b - [6a - \{3b - (6a - 6b)\}]$.
2. $2x - \{3y - [4x - (5y - 6x)]\}$.
3. $\{50 - [35 - (10 - x)x]x\}x$.
4. $4x - \{[x - 3(2 - x)]x - 4\}2$.
5. $6a - [7a - \{8a - (9a - \overline{10a - b})\}]$.
6. $a - \{5b - [a - (3c - 3b) + 2c - (a - 2b - c)]\}$.
7. $12a - 13\{10[7(4a - 3) - 6] - 9a\}$.
8. $x - \{x + y - [x + y + z - (x + y + z + v)]\}$.
9. $10 - 2\{x - 5[3 - 2x - 6(4x - 7)] - 3(5 - 2x)\}$.
10. $7a^m - \{2a^m - [a^n - 3a^m + (5a^m - 2a^n) - 4a^m] - 2a^n\}$.
11. $\{[(x + y^2)x - (2y - 1)]x - (x^2 - 2y)x - x^2y^2\}^2$.
12. $[(x - y)^2 + 6xy] - [(x^2 + 2xy) - \{x^2 - [2xy - (4xy - y^2)]\}$
    $- (-x^2 - 2xy)]$.
13. $\frac{2}{3}a^2x + \frac{3}{4}ax^2 - \{\frac{1}{4}ax^2 - [-\frac{2}{3}a^2x - (\frac{1}{2}ax^2 - a)]\}$.

Find the values of the expressions in Exx. 1–13,

14. When $a = -3$, $b = 4$, $c = -5$, $m = 2$, $n = 1$, $x = 8$, $y = -9$, $z = 7$, $v = -4$.

Solve the following equations:

15. $(x + 3)^2 = x^2 + 15$.
16. $(2x + 1)^2 - 8 = (2x - 1)^2$.
17. $(5 + x)^2 + 9 = 3x(9 + \frac{1}{3}x)$.
18. $(x + 1)^2 = [6 - (1 - x)]x - 2$.
19. $(3 - \frac{1}{2}x)^2 + (\frac{2}{3}x - 5)^2 = (\frac{5}{6}x)^2$.
20. $2x^2 + 17x = (8 + 2x)^2 - 67 - x(3 + 2x)$.
21. $(2x^2 + 1 - x)(2x^2 - 1 + x) = 1 + x^2(2x + 1)(2x - 1)$.

22. $(x-1)(x^2+x+1)-6(x^2-1)=-(2-x)^3.$

23. $56x^3-4x(4x-7)^2=164x^2-(2x-5)^3.$

24. $6-\{5-(4-\{3-[2-(1-x)]\})\}=4.$

25. $2[8-2\{6-2(5-\overline{2x-1})\}]=8.$

26. $\frac{2}{3}[\frac{3}{2}\{\frac{3}{2}(6-x)-x\}-x]-x=4.$

27. $4\{4[4(4x-3)-3]-3\}-3=1.$

28. $-4-4\{4-4[4-4(4-x)]\}=44.$

29. $-4x-(5x-[6x-\{7x-(8x-9)\}])=-10.$

Insertion of Parentheses.

5. The principles for inserting parentheses in a given expression were proved in Ch. II., § 2, Art. 9.

Ex. 1. Express $4(x-y)+y-x$ as a product, of which one factor is $x-y$,

We have $4(x-y)+y-x=4(x-y)-(x-y)=3(x-y)$.

The sign + or − before a pair of parentheses can evidently be reversed from + to −, or from − to +, if the signs of the terms within the parentheses be reversed.

Ex. 2. $7(x-1)-3(1-x)=7(x-1)+3(x-1)=10(x-1)$.

EXERCISES II.

Write each of the following expressions as a product, of which the expression within the parentheses is one of the factors:

1. $3(a-b)-a+b.$ **2.** $5(x^2-y)-x^2+y.$

3. $3m-5n-4(5n-3m).$ **4.** $1-a^n+3(a^n-1).$

5. $5(x^2-x+1)-x^2+x-1.$ **6.** $x-y-z-6(y+z-x).$

Write each of the following expressions as a single product, of which the expression within the first parentheses is a factor:

7. $(2x-1)-3(1-2x).$ **8.** $2(2m-3n)+(3n-2m).$

9. $5(x^2-y^2)+2(y^2-x^2).$ **10.** $7(xy-z)-(z-xy).$

Simplify the following expressions without removing the parentheses:

11. $(a-b)c+(b-a)c.$ **12.** $5(x-y)z+5(y-x)z.$

13. $(1-x)(1+x^2)+(x-1)(1+x^2).$

14. $9(xy+3)(z-5)+7(xy+3)(5-z).$

CHAPTER VIII.

FACTORS AND MULTIPLES OF INTEGRAL ALGEBRAIC EXPRESSIONS.

§ 1. INTEGRAL ALGEBRAIC FACTORS.

1. Factors have already been defined in multiplication (Ch. II., § 3, Art. 12). The factors were there given, and their product was required. The converse process, given a product to find its factors, is equally important.

2. A product of two or more factors is, by the definition of division, exactly divisible by any one of the factors.

An **Integral Algebraic Factor** of an expression is an integral expression by which the given one is exactly divisible.

E.g., integral factors of $6a^2x$ are 6, a^2x, $3x$, $2a^2$, etc.;
integral factors of $a^2 - b^2$ are $a + b$ and $a - b$.

The word *integral*, here as in Ch. III., § 1, Art. 1, refers only to the *literal* parts of the expression.

E.g., $\frac{1}{2}a$ and $\frac{3}{4}x$ are integral algebraic factors of $6a^2x$.

3. A **Prime Factor** is one which is exactly divisible only by itself and unity.

E.g., the prime factors of $6a^2x$ are $2, 3, a, a, x$.

A **Composite Factor** is one which is not prime, *i.e.*, which is itself the product of two or more prime factors.

E.g., composite factors of $6a^2x$ are 6, ax, $2a$, $3ax$, etc.

4. Any monomial can be resolved into its prime factors by inspection.

E.g., the prime factors of $4a^3b^2$ are $2, 2, a, a, a, b, b$.

The Fundamental Formula for Factoring.

5. A multinomial whose terms contain a common factor can be factored by applying the converse of the Distributive Law for Multiplication. From Ch. III., § 3, Art. 14, we have

$$\boldsymbol{ab + ac - ad = a(b + c - d)}. \qquad (1)$$

That is, *if the terms of a multinomial contain a common factor, the multinomial can be written as the product of the common factor and the algebraic sum of the remaining factors of the terms.*

The relation (1) may be called the *Fundamental Formula for Factoring.*

Ex. 1. Factor $2x^2y - 2xy^2$.

The factor $2xy$ is common to both terms; the remaining factor of the first term is x, that of the second term is $-y$, and their algebraic sum is $x - y$.

Consequently $2x^2y - 2xy^2 = 2xy(x - y)$.

Ex. 2. $ab^2 + abc + b^2c = b(ab + ac + bc)$.

6. In the fundamental formula the letters a, b, c, d may stand for binomial or multinomial expressions.

Ex. 1. Factor $a(x - 2y) + b(x - 2y)$.

The factor $x - 2y$ is common to both terms; the remaining factor of the first term is a, that of the second term is b, and their algebraic sum is $a + b$.

Consequently $a(x - 2y) + b(x - 2y) = (x - 2y)(a + b)$.

Ex. 2. Factor $1 - a + x(1 - a)$.

We have $1 - a + x(1 - a) = (1 - a)(1 + x)$.

Ex. 3. Factor $(x - y)(a^2 + b^2) - (x + y)(a^2 + b^2)$.

We have

$$\begin{aligned}(x - y)(a^2 + b^2) - (x + y)(a^2 + b^2) &= (a^2 + b^2)[(x - y) - (x + y)] \\ &= -2y(a^2 + b^2).\end{aligned}$$

It frequently happens that the parts of a given expression have a common factor except for sign.

Ex. 4. Factor $x^2(1-m)-y^2(m-1)$.

Since $1-m$ and $m-1$ differ only in sign, *i.e.*,

$$m-1=-(1-m),$$

we may take either as the common factor.

Taking $1-m$ as the common factor, we have

$$x^2(1-m)-y^2(m-1)=(1-m)(x^2+y^2).$$

The fundamental formula must often be applied more than once.

Ex. 5. Factor $by(x-a)-bx(y-a)$.

Taking out b, $b[y(x-a)-x(y-a)]=b(ax-ay)$.

Taking out a, we have $ab(x-y)$.

EXERCISES I.

Factor the following expressions:

1. $5x+5$.
2. $ax-a$.
3. $4a^3-6$.
4. $-x^3-x^2$.
5. a^2b-ab^2.
6. $2an-4n^2$.
7. $3x^4-2x^3$.
8. $12a^3b^3-3a^2b^2$.
9. $10a^4x^2-15a^2x^4$.
10. $3ab+6ac-12ad$.
11. $70xy-98y^2-140yz$.
12. $\frac{15}{16}ax+\frac{15}{16}bx^2+\frac{1}{4}x$.
13. $6ax^4-15a^3bx^5+18a^2b^2x^6$.
14. $8a^2n^5x^5-10an^4x^7+4a^2n^3x^8$.
15. $45m^3n^3p+90m^3n^2p-75m^2np^2$.
16. $28a^5b^3c-84a^3b^4c^2+98a^4b^4c^3$.
17. $27x^3y^4z^2+135x^5y^4z^4-81x^4y^4z^4$.
18. $7(a+b)-14$.
19. $a^2(a+x)+x^2(a+x)$.
20. $3a(a-1)-3(a-1)$.
21. $2(n+1)^2-4(n+1)$.
22. $a(x-1)-x+1$.
23. $m(q-p)-(p-q)$.
24. $-2a^x+4a^{2x}+6a^{3x}$.
25. $a^{n+1}-a+a^{n-1}$.
26. $6m^{n+1}-3m^{n+2}+9m^{n+3}$.
27. $5^{n+3}-125x+625x^2$.
28. $ax^n-bx^{n+1}+cx^{n+2}$.
29. $2^{n+4}-8\times 2^{n-1}+16$.

Grouping Terms.

7. When all the terms of a given expression do not contain a common factor, it is sometimes possible to group the terms so that all the groups shall contain a common factor.

Ex. 1. Factor $2a + 2b + ax + bx$.

Factoring the first two terms by themselves, and the last two terms by themselves, we obtain

$$2(a+b) + x(a+b) = (a+b)(2+x).$$

Ex. 2. $x^2 - xy - xz + yz = x(x-y) - z(x-y) = (x-y)(x-z)$.

Ex. 3. $(2a-b)^2 + 4ax - 2bx = (2a-b)^2 + 2x(2a-b)$

$$= (2a-b)[2a-b+2x].$$

EXERCISES II.

Factor the following expressions:

1. $ac + ad + bc + bd$.

2. $2ax - 3by - 2ay + 3bx$.

3. $20ad - 35bd - 8ax + 14bx$.

4. $5ax - cx - 5ay + cy$.

5. $a^3 - a^2c + ac^2 - c^3$.

6. $x^3 - x^2 + x - 1$.

7. $18n^2x - 12x - 9n^2 + 6$.

8. $3x^4 - x^3 + 6x - 2$.

9. $3c^4 - 3c^3n + cn^2 - n^3$.

10. $3x^3 + nx^2 - 6n^2x - 2x^3$.

11. $6a^3 - 6a^2y + 2ay^2 - 2y^3$.

12. $\frac{1}{6}ty + \frac{2}{3}xy - \frac{1}{8}tz - \frac{1}{2}xz$.

13. $12a^3b^4 - 4a^2b^4 - 4a^2b^3 + 12a^3b^3$.

14. $a^4 - a^3n^2 + a^2n - an^3 + n^5 - an^3$.

15. $x^4 - ax^3 + 3a^2x^2 - 2a^2bx^2 + 2a^3bx - 6a^4b$.

16. $ax + by + cz + bx + cy + az + cx + ay + bz$.

17. $ax - by + cz - bx - cy - az - cx + ay + bz$.

18. $ax + by + cz - bx - cy + az + cx - ay - bz$.

19. $ax + by + cz - bx + cy - az - cx - ay + bz$.

20. $6x^n + 8x^{n-1} - 9x^{n-2} - 12x^{n-3} + 3x^{n-4} + 4x^{n-5}$.

Use of Type-Forms in Factoring.

8. If an expression be in the form of one of the type-forms considered in Ch. VI., or if it can be reduced to such a form, its factors can be written by inspection.

Trinomial Type-Forms.

9. From Ch. VI., § 1, Arts. 1 and 2, we have

$$a^2 + 2ab + b^2 = (a+b)^2,$$

$$a^2 - 2ab + b^2 = (a-b)^2.$$

From these identities we see that a trinomial which is the square of a binomial must satisfy the following conditions:

(i.) *One term of the trinomial is the square of the first term of the binomial.*

(ii.) *A second term of the trinomial is the square of the second term of the binomial.*

(iii.) *The remaining term of the trinomial is twice the product of the two terms of the binomial.*

Ex. 1. Factor $x^2 + 6x + 9$.

x^2 is the square of x, 9 is the square of 3, and $6x = 2 \cdot x \cdot 3$.

Therefore $$x^2 + 6x + 9 = (x + 3)^2.$$

Ex. 2. Factor $-4xy + 4x^2 + y^2$.

$4x^2$ is the square of $2x$, or of $-2x$; y^2 is the square of y, or of $-y$. Since the middle term in the given expression is negative, one term of the binomial is negative, the other positive.

Therefore $-4xy + 4x^2 + y^2 = (2x - y)^2 = (-2x + y)^2$.

Ex. 3. $$\begin{aligned} 60xy - 36x^2 - 25y^2 &= -(36x^2 - 60xy + 25y^2) \\ &= -(6x - 5y)^2. \end{aligned}$$

Ex. 4. $$\begin{aligned} &(n^2 - 2nx)^2 + 2(n^2x^2 - 2nx^3) + x^4 \\ &= (n^2 - 2nx)^2 + 2x^2(n^2 - 2nx) + x^4 \\ &= (n^2 - 2nx + x^2)^2 = [(n - x)^2]^2 = (n - x)^4. \end{aligned}$$

EXERCISES III.

Factor the following expressions:

1. $x^2 - 2x + 1$.
2. $a^2 + 6a + 9$.
3. $y^2 + 12y + 36$.
4. $a^2 - 10a + 25$.
5. $4x^2 - 12x + 9$.
6. $9a^2 + 30a + 25$.
7. $20x - 4x^2 - 25$.
8. $36x - 4x^2 - 81$.
9. $16a^2 + 40ab + 25b^2$.
10. $49x^2 - 28xy + 4y^2$.
11. $9x^2y^2 - 30xyz^2 + 25z^4$.
12. $24xy - 9x^2 - 16y^2$.
13. $a^4 - 2a^2x + x^2$.
14. $x^4 - 2x^2y^2 + y^4$.
15. $a^2x^2 - 4ac^3x + 4c^6$.
16. $2a^2x^2 - a^4 - x^4$.
17. $(a + x)^2 + 2(a + x) + 1$.
18. $(x - 4)^2 - 4(x - 4) + 4$.

19. $(2x-9)^2-6(9-2x)+9$.

20. $4x^{2n}-12x^n+9$.

21. $36a^{n+2}-48a^n+16a^{n-2}$.

22. $4ax+2a^2+2x^2$.

23. $6a^2x^2-3a^2x^3-3a^2x$.

24. $16a^2b^6+9c^8+24ab^3c^4$.

25. $a^{2n-2}-2a^{n-1}x^{n+1}+x^{2n+2}$.

26. $(a^2+2ab+b^2)c+(a+b)d^2$.

27. $xy-xz-(y^2-2yz+z^2)$.

28. $a^2+2an+n^2-ap-pn$.

29. $2a+ad-d^2-4d-4$.

30. $a^2+2ab-4ac-4bc+4c^2$.

31. $n^2x-xy-n^4y+2n^2y^2-y^3$.

32. $(a-c)^3+2a^2c-4ac^2+2c^3$.

10. From Ch. VI., § 1, Art. 5, we have

$$\boldsymbol{x^2+(a+b)x+ab=(x+a)(x+b).}$$

When a trinomial, arranged to descending powers of some letter, say x, can be factored into two binomials, in both of which the first term is the letter of arrangement, it must satisfy the following conditions:

(i.) *One term of the trinomial is the square of the letter of arrangement*, i.e., *of the common first term of the binomial factors.*

(ii.) *The coefficient of the first power of the letter of arrangement in the trinomial is the algebraic sum of two numbers whose product is the remaining term of the trinomial.*

(iii.) *These two numbers are the second terms of the binomial factors.*

Ex. 1. Factor $x^2+8x+15$.

The common first term of the binomial factors is evidently x. The second terms are two numbers whose product is 15, and whose sum is 8. By inspection we see that

$$3+5=8 \text{ and } 3\times 5=15;$$

that is, the second terms of the binomial factors are 3 and 5.

Consequently, $x^2+8x+15=(x+3)(x+5)$.

Ex. 2. Factor $x^2-7x+12$.

The common first term of the binomial factors is x. The second terms are two numbers whose product is 12, and whose sum is -7. Since their product is *positive*, they must be *both positive* or *both negative;* and since their sum is negative, they must be *both negative.*

The possible pairs of negative factors of 12 are -1 and -12, -2 and -6, -3 and -4.

But since $-3+(-4)=-7$,

the second terms of the binomial factors are -3 and -4.

Consequently $x^2-7x+12=(x-3)(x-4)$.

Ex. 3. Factor $a^2x^2+5ax-24$.

The common first term of the binomial factors is ax. The second terms are two numbers whose product is -24, and whose sum is 5. Since their product is negative, one must be positive and the other negative; and since their sum is positive, the positive number must have the greater absolute value. The possible pairs of factors of -24 are -1 and 24, -2 and 12, -3 and 8, -4 and 6.

But since $-3+8=5$,

the second terms of the binomial factors are -3 and 8.

Consequently $a^2x^2+5ax-24=(ax-3)(ax+8)$.

Ex. 4. Factor $x^2-3xy-28y^2$.

The common first term of the binomial factors is x. The second terms are two numbers whose product is $-28y^2$, and whose sum is $-3y$. It is evident that both of these terms contain y as a factor. Therefore we have only to find their numerical coefficients.

Since their product is negative, one must be positive and the other negative; and since their sum is negative, the negative number must have the greater absolute value. The possible pairs of factors of -28 are 1 and -28, 2 and -14, 4 and -7.

But since $4+(-7)=-3$,

the second terms of the binomial factors are $4y$ and $-7y$.

Consequently $x^2-3xy-28y^2=(x+4y)(x-7y)$.

EXERCISES IV.

Factor the following expressions:

1. x^2-3x+2.
2. x^2-x-2.
3. x^2+x-6.
4. x^2+3x+2.
5. x^2+x-2.
6. x^2-x-6.
7. x^2-5x+6.
8. x^2-6x+5.
9. $x^2-4x-60$.

10. $x^2 + 7x - 30$. **11.** $x^2 + 12x + 32$. **12.** $x^2 - 3x - 40$.

13. $x^2 - 12x + 35$. **14.** $x^3 - 17x^2 + 72x$. **15.** $x^2 + 13x - 30$.

16. $6x - x^2 - x^3$. **17.** $35 + 2x - x^2$. **18.** $x^4 - 5x^2 - 24$.

19. $x^4 + 8x^2 + 15$. **20.** $x^4 - 24x^2 + 63$. **21.** $3x^6 + 39x^3 + 66$.

22. $x^6 - x^3 - 56$. **23.** $x^{2n} + 6x^n - 112$. **24.** $x^{2n} - 16x^n + 55$.

25. $x^2 + (a + b)x + ab$. **26.** $x^2 - (m + n)x + mn$.

27. $x^2 + (p - q)x - pq$. **28.** $x^2 + (3r - 2s)x - 6rs$.

29. $ax^2 + 7a^2x + 6a^3$. **30.** $x^2 + 2xy - 15y^2$.

31. $x^2 - 4ax - 12a^2$. **32.** $x^2 - 7ax + 12a^2$.

33. $2ax^3y^2 - 26ax^2y^3 + 84axy^4$. **34.** $x^2 - 11xm + 30m^2$.

35. $x^2z^2 + 12xz - 13$. **36.** $a^2b^2 - 7ab + 10$.

37. $m^2n^2 - 20mn + 99$. **38.** $(a + b)^2 + 7(a + b) + 6$.

39. $(a - b)^2 + 7(a - b) + 12$. **40.** $(m + n)^2 + 2(m + n) - 15$.

11. From Ch. VI., § 1, Art. 6, we have

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

A trinomial which can be factored by this type-form must satisfy the following conditions:

(i.) *One term of the trinomial is the product of the first terms of its binomial factors.*

(ii.) *A second term of the trinomial is the product of the second terms of its binomial factors.*

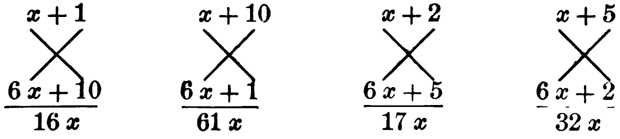
(iii.) *The remaining term of the trinomial is the sum of the products of the first term of each binomial factor by the second term of the other.*

Ex. 1. Factor $6x^2 + 19x + 10$.

The first terms of the required binomial factors are factors of $6x^2$, the second terms are factors of 10, and the sum of the cross-products is $19x$.

The factors of $6x^2$ are x and $6x$, $2x$ and $3x$; and the factors of 10 are 1 and 10, 2 and 5.

The following arrangements represent possible pairs of factors:



$$\begin{array}{c} 2x+1 \\ \times \\ \underline{3x+10} \\ 23x \end{array} \qquad \begin{array}{c} 2x+10 \\ \times \\ \underline{3x+1} \\ 32x \end{array} \qquad \begin{array}{c} 2x+2 \\ \times \\ \underline{3x+5} \\ 16x \end{array} \qquad \begin{array}{c} 2x+5 \\ \times \\ \underline{3x+2} \\ 19x \end{array}$$

Since the sum of the cross-products in the last arrangement is equal to the middle term of the given trinomial, we have

$$6x^2 + 19x + 10 = (2x+5)(3x+2).$$

Ex. 2. Factor $5x^2 - 6xy - 8y^2$.

The factors of $5x^2$ are x and $5x$, and the factors of $-8y^2$ are y and $-8y$, $-y$ and $8y$, $2y$ and $-4y$, $-2y$ and $4y$.

$$\begin{array}{c} x-2y \\ \times \\ \underline{5x+4y} \\ -6xy \end{array}$$

Since the sum of the cross-products in the arrangement on the left is equal to the middle term of the given trinomial, we have

$$5x^2 - 6xy - 8y^2 = (x-2y)(5x+4y).$$

Observe that the reason given in Art. 10, Ex. 4, for rejecting at sight some factors of the last term of the trinomial does *not* hold in the above example. For, although the middle term, $-6xy$, is negative, the negative factor of $-8y^2$ is less in absolute value than the positive factor.

Ex. 3. Factor $10a^4 + a^2b - 21b^2$.

The factors of $10a^4$ are a^2 and $10a^2$, $2a^2$ and $5a^2$; and the factors of $-21b^2$ are b and $-21b$, $-b$ and $21b$, $3b$ and $-7b$, $-3b$ and $7b$.

$$\begin{array}{c} 2a^2+3b \\ \times \\ \underline{5a^2-7b} \\ a^2b \end{array}$$

Since the sum of the cross-products in the arrangement on the left is equal to the middle term of the given trinomial, we have

$$10a^4 + a^2b - 21b^2 = (2a^2+3b)(5a^2-7b).$$

Ex. 4. Factor $-15x^2 + 22x - 8$.

We have
$$\begin{aligned} -15x^2 + 22x - 8 &= -(15x^2 - 22x + 8) \\ &= -(3x-2)(5x-4) \\ &= (2-3x)(5x-4). \end{aligned}$$

12. The following directions may be observed in factoring trinomials which come under this type-form:

(i.) *When all the terms of the trinomial are positive, only positive factors of the last term are to be tried.*

(ii.) *When the middle term is negative and the last term is positive, the factors of the last term must be both negative.*

(iii.) *When the middle term and the last term are both negative, one factor of the last term must be positive, the other negative.*

(iv.) *Select that pair of factors of the last term which, by cross-multiplication, gives the middle term of the trinomial.*

EXERCISES V.

Factor the following expressions:

1. $6x^2 + x - 12.$
2. $6x^2 - x - 12.$
3. $35x^2 + 32x - 12.$
4. $35x^2 + x - 12.$
5. $35x^2 + 16x - 12.$
6. $35x^2 - 13x - 12.$
7. $2x^2 + 5x + 2.$
8. $10 + 16x + 6x^2.$
9. $6 + 13x - 63x^2.$
10. $3x^2 + 13x + 12.$
11. $40 + 2x - 2x^2.$
12. $25x^3 + 25x^2 - 6x.$
13. $36x^4 - 18x^2 - 10.$
14. $12x - 6x^2 - 90x^3.$
15. $10x^2 + 7x - 33.$
16. $8x^4 - 19x^2 - 15.$
17. $40 + 6x - 27x^2.$
18. $49x^2 - 35x + 6.$
19. $64x^2 - 92x + 30.$
20. $6 - 19x + 15x^2.$
21. $6x^2 - 41x - 56.$
22. $30x^2 - 89x + 35.$
23. $18x^2 - 3xy - 45y^2.$
24. $3a^2 - 5ab - 2b^2.$
25. $18x^4 + 3x^2y - 10y^2.$
26. $abx^2 - (a^2 - b^2)x - ab.$
27. $5a^4x^2 - 4a^2xz - 96z^2.$
28. $-10a^4 + 7a^2b^2 + 12b^4.$
29. $4x^2 - xy - 3y^2.$
30. $10a^2 + 11ab - 6b^2.$
31. $9x^{2n} - 4x^n - 5.$
32. $2x^{2r+2} - 3x^{r+1} - 2.$
33. $6x^{2m} + x^my^n - 15y^{2n}.$
34. $10(a + b)^2 + 7c(a + b) - 6c^2.$
35. $7(x - y)^2 - 37z(x - y) + 10z^2.$
36. $6(x^2 + y^2)^2 - 9(x^2 + y^2)z^2 - 15z^4.$
37. $2(a^2 - c^2)^2 - 4b(a^2 - c^2) - 6b^2.$

Binomial Type-Forms.

13. From Ch. VI., § 1, Art. 4, we have

$$a^2 - b^2 = (a + b)(a - b).$$

That is, *the difference of the squares of two numbers can be written as the product of the sum and the difference of the numbers.*

Ex. 1.
$$\begin{aligned} a^2x^2 - \tfrac{1}{4}b^2 &= (ax)^2 - (\tfrac{1}{2}b)^2 \\ &= (ax + \tfrac{1}{2}b)(ax - \tfrac{1}{2}b). \end{aligned}$$

Ex. 2.
$$\begin{aligned}32\,m^4n-2\,n^3&=2\,n(16\,m^4-n^2)\\&=2\,n[(4\,m^2)^2-n^2]\\&=2\,n(4\,m^2+n)(4\,m^2-n).\end{aligned}$$

Ex. 3.
$$\begin{aligned}x^2-4\,xy+4\,y^2-9\,z^2&=(x-2\,y)^2-(3\,z)^2\\&=(x-2\,y+3\,z)(x-2\,y-3\,z).\end{aligned}$$

In factoring a given expression the type-form must frequently be applied more than once.

Ex. 4.
$$\begin{aligned}&4\,a^2c^2-(a^2-b^2+c^2)^2\\&\quad=(2\,ac+a^2-b^2+c^2)(2\,ac-a^2+b^2-c^2)\\&\quad=[(a+c)^2-b^2][b^2-(a-c)^2]\\&\quad=(a+c+b)(a+c-b)(b+a-c)(b-a+c).\end{aligned}$$

14. The difference of the like, or unlike, even powers of two numbers can always be written as the difference of the squares of two numbers, and should therefore first be factored by applying this type-form.

Ex.
$$\begin{aligned}a^4-b^4&=(a^2)^2-(b^2)^2\\&=(a^2+b^2)(a^2-b^2)\\&=(a^2+b^2)(a+b)(a-b).\end{aligned}$$

EXERCISES VI.

Factor the following expressions:

1. x^2-1. **2.** $4-a^2$. **3.** $a^2-x^2y^2$.
4. $25\,x^2-9$. **5.** $36\,a^2-49\,b^2$. **6.** $4\,x^8-y^4$.
7. 86^2-14^2. **8.** 57^2-43^2. **9.** 87^2-27^2.
10. $81\,a^4-16$. **11.** $\frac{4}{9}\,a^2b^2-\frac{25}{49}\,c^2d^2$. **12.** $16\,a^6-25\,b^4c^6$.
13. $a^2b^4c^6-\frac{1}{4}$. **14.** $\frac{1}{9}\,a^2n^4-\frac{1}{100}\,x^6$. **15.** $a^{2n}-1$.
16. $a^{2n}-b^{2m}$. **17.** $x^{2r+2}-4$. **18.** $9\,a^{2n}b^2-4\,c^{2m}$.
19. $(m-n)^2-1$. **20.** $c^2-(a-b)^2$. **21.** $9-(3-x)^2$.
22. $(4\,x-3)^2-16\,x^2$. **23.** $(a-b)^2-(c-d)^2$.
24. $(5\,x-2)^2-(4\,x-3)^2$. **25.** $(3\,xy-5)^2-(2\,xy-6)^2$.
26. $(x^2+x+1)^2-(x^2-x+1)^2$. **27.** a^4-a^3+a-1.
28. $7-112\,x^4$. **29.** $16\,x^4-y^4$. **30.** a^8-b^8.
31. $1-256\,x^8y^8$. **32.** $x^{16}-y^{16}$. **33.** $a^{16}-1$.
34. $5\,a^2-180\,b^4$. **35.** $75\,a^2b^4-108\,c^2d^4$. **36.** $243\,b^5c^6-75\,b^7$.

37. $\frac{2}{4} ab^2 - \frac{8}{9} ac^4$. **38.** $\frac{5}{4} xy^4 - \frac{5}{25} xz^6$. **39.** $a^{2n} - b^{2m}$.

40. $a^{4x} - b^{4x}$. **41.** $144 x^n - x^{n+2}$. **42.** $4 a^{3n+3} - a^{n+1}$.

43. $a^{2n+3}b^{2n} - a^5b^{2n+2}$. **44.** $m^{2n-4}n^{6m+2} - 1$. **45.** $a^2 - b^2 + (a + b)c$

46. $x^2 + 3x^3 - x^4 - 3x$. **47.** $a^2 - x^2 + a - x$. **48.** $x^2 - xz - yz - y^2$.

49. $a^2 - a^2n + an^2 - n^2$. **50.** $x^2 - 2xy + y^2 - z^2$. **51.** $a^2 + 2bc - b^2 - c^2$

52. $a^2 - n^2 + 2np - p^2$. **53.** $p^2 - z^2 - 4z - 4$.

54. $a^4 - 2ab^3 - b^4 + 2a^3b$. **55.** $a^2 + b^2 - c^2 - d^2 + 2(ab + cd)$.

56. $x^3y - xy^3 + x^2y + xy^2$. **57.** $a^2 + b^2 - c^2 - d^2 - 2(ab - cd)$.

58. $2(ab + cd) - (a^2 + b^2 - c^2 - d^2)$. **59.** $a^2 - b^2 + 2bz - 2ax + x^2 - z^2$.

60. $4a^2b^2 - (a^2 + b^2 - c^2)^2$. **61.** $a^{2r} - a^{4r} - 2a^{7r} - a^{10r}$.

62. $a^4 + 4a^2c - 4b^2 + 4bd + 4c^2 - d^2$. **63.** $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.

64. $(a + n)(a^2 - x^2) - (a - x)(a^2 - n^2)$.

65. $(n - x)(5n^2 - 4x^2) - (3x^2 - 4n^2)(x - n)$.

66. $(a + b)^2 - 1 - 2(a + b + 1)$.

67. $(a - 2b)^2 - 9 - 3(a - 2b + 3)$.

15. From Ch. VI., § 2, Art. 2 (i.) and (ii.), we derive

$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2),}$$
$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2).}$$

Ex. **1.**

$$\begin{aligned} x^3 + 8y^3 &= x^3 + (2y)^3 \\ &= (x + 2y)[x^2 - x(2y) + (2y)^2] \\ &= (x + 2y)(x^2 - 2xy + 4y^2). \end{aligned}$$

Ex. **2.**

$$\begin{aligned} (1 - x)^3 - 8x^3 &= (1 - x)^3 - (2x)^3 \\ &= (1 - x - 2x)[(1 - x)^2 + (1 - x)(2x) + (2x)^2] \\ &= (1 - 3x)(1 + 3x^2). \end{aligned}$$

Ex. **3.**

$$\begin{aligned} 512x^9 + y^9 &= (8x^3)^3 + (y^3)^3 \\ &= (8x^3 + y^3)[(8x^3)^2 - (8x^3)(y^3) + (y^3)^2] \\ &= [(2x)^3 + y^3](64x^6 - 8x^3y^3 + y^6) \\ &= (2x + y)(4x^2 - 2xy + y^2)(64x^6 - 8x^3y^3 + y^6). \end{aligned}$$

Ex. **4.**

$$\begin{aligned} a^6 - 729b^6 &= (a^3)^2 - (27b^3)^2 \\ &= (a^3 + 27b^3)(a^3 - 27b^3) \\ &= (a + 3b)(a^2 - 3ab + 9b^2)(a - 3b)(a^2 + 3ab + 9b^2). \end{aligned}$$

16. From Ch. VI., § 2, Art. 2 (i.) and (ii.), we infer:

(i.) *The sum of the like odd powers of two numbers contains the sum of the numbers as a factor.*

(ii.) *The difference of the like odd powers of two numbers contains the difference of the numbers as a factor.*

Ex. 1. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

Ex. 2. $x^7 - y^7 = (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)$.

17. The sum of the like even powers of two numbers, whose exponents are divisible by an odd number, except 1, can be factored by applying the type-forms of Arts. 15 and 16.

Ex.
$$\begin{aligned} x^{12} + y^{12} &= (x^4)^3 + (y^4)^3 \\ &= (x^4 + y^4)[(x^4)^2 - (x^4)(y^4) + (y^4)^2] \\ &= (x^4 + y^4)(x^8 - x^4y^4 + y^8). \end{aligned}$$

EXERCISES VII.

Factor the following expressions:

1. $x^3 + 1$.
2. $x^3 - 8$.
3. $a^3 + 27$.
4. $64x^3 - 1$.
5. $8x^3 - y^6$.
6. $8x^3y^3 - 27$.
7. $125x^3y^6 + 8$.
8. $3a^2 - 24a^5$.
9. $27a - a^4b^6$.
10. $2x^3y^5 + 432y^2$.
11. $a^5 + 243$.
12: $x^6 + y^6$.
13. $x^6 - 64$.
14. $x^9 + y^9$.
15. $x^9 - 1$.
16. $27x^3 - y^9$.
17. $a^9b^9 + 64c^6$.
18. $125x^3 - y^{12}z^{12}$.
19. $a^{10} - b^{10}$.
20. $x^{10} + y^{10}$.
21. $x^{12} - 1$.
22. $x^{14}y^{14} - 1$.
23. $x^{14} + y^{14}$.
24. $a^{15} + b^{15}$.
25. $1 - x^{15}$.
26. $1 - a^{18}$.
27. $x^{18} + y^{18}$.
28. $a^{3n} - b^{3n}$.
29. $8x^{3n}y^n - 729y^{n+3}z^6$.
30. $27 - (3 + 2x)^3$.
31. $(2a + x)^3 + (a - 2x)^3$.
32. $4 - x^2 + 4x^3 - x^5$.
33. $x^3 - y^3 - 2x^2y + 2xy^2$.
34. $(a + b)^3 - (c + d)^3$.
35. $x^5 - x^3 - x^2 + 1$.
36. $x^3 - 8 - 6x^2 + 12x$.
37. $a^3 - 4a^2c - 4ac^2 + c^3$.
38. $n^6 + 5n^4x^2 + 5n^2x^4 + x^6$.

18. The method of Art. 11 can be extended to factor multinomials of the second degree whose factors contain three or more terms.

Ex. 1. Factor $ax^2 - ay^2 + x + y - ax + ay - 1$.

The terms $ax^2 - ay^2$ are evidently the product of the terms which contain x and y in the two factors. These may be either

$$ax + ay \text{ and } x - y, \text{ or } ax - ay \text{ and } x + y.$$

Since the last term of the multinomial does not contain either x or y, it must be the product of terms in the factors which do not contain x or y. These can be only $+1$ and -1. The following arrangements represent possible pairs of factors:

$$\begin{array}{r} (ax+ay)+1 \\ \times \\ (x-y)-1 \\ \hline x-y-ax-ay \end{array} \qquad \begin{array}{r} (ax+ay)-1 \\ \times \\ (x-y)+1 \\ \hline -x+y+ax+ay \end{array} \qquad \begin{array}{r} (ax-ay)+1 \\ \times \\ (x+y)-1 \\ \hline x+y-ax+ay \end{array} \qquad \begin{array}{r} (ax-ay)-1 \\ \times \\ (x+y)+1 \\ \hline -x-y+ax-ay \end{array}$$

Since the third arrangement gives the remaining terms, $x+y-ax+ay$, of the multinomial, we have

$$ax^2-ay^2+x+y-ax+ay-1=(x+y-1)(ax-ay+1).$$

Ex. 2. Factor $2x^2-12y^2+4z^2-5xy-8yz-9xz$.

The given expression suggests the product of two trinomials, both of which contain terms in x, y, and z. The part $2x^2-5xy-12y^2$ is evidently the product of the parts of the factors which contain terms in x and y, and the term $4z^2$ is evidently the product of the terms in z in the factors.

The factors of $2x^2-5xy-12y^2$ are found to be $2x+3y$ and $x-4y$; the factors of $4z^2$ are z and $4z$, $-z$ and $-4z$, $2z$ and $2z$, $-2z$ and $-2z$. By trial we find that the following arrangement

$$\begin{array}{r} (2x+3y)-\ z \\ \times \\ (x-4y)-4z \\ \hline -9xz-8yz \end{array}$$

gives the remaining terms of the multinomial. Consequently,

$$2x^2-12y^2+4z^2-5xy-8yz-9xz=(2x+3y-z)(x-4y-4z).$$

EXERCISES VIII.

Factor the following expressions:

1. x^2-y^2-2y-1.
2. $a^2+b^2+2ab+8a+8b-9$.
3. $x^2+2xy+3x+y^2+3y+2$.
4. $2x^2-3y^2-z^2+xy+xz+4yz$.
5. $x^2+3xy+5x+2y^2+8y+6$.
6. $2a^2-9ac-5ab+4c^2-8bc-12b^2$.
7. $x^2-3xy+3x+2y^2-5y+2$.
8. $x^2+4y^2-4xy-10x+20y-56$.

9. $4x^2+9y^2+19-12xy+40x-60y$.

10. $56x^2-6y^2-12z^2-5xy+34xz-22yz$.

11. $3x^2-4ax+2bx-2ab+a^2$.

Special Devices for Factoring.

19. A factorable expression can frequently be brought to some known type-form by adding to, or subtracting from it one or more terms.

Ex. 1. Factor $x^4+x^2y^2+y^4$.

This expression would be the square of x^2+y^2, if the coefficient of x^2y^2 were 2. We therefore add x^2y^2, and, in order that the value of the expression may remain the same, we subtract x^2y^2. We then have

$$\begin{aligned} x^4+2x^2y^2+y^4-x^2y^2 &= (x^2+y^2)^2-x^2y^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy). \end{aligned}$$

Ex. 2. Factor x^3+3x^2-2.

The terms x^3+3x^2 suggest the cube of $x+1$.

To complete the cube of $x+1$ we must add, and therefore also subtract, $3x+1$. We then obtain

$$\begin{aligned} x^3+3x^2+3x+1-3x-3 &= (x+1)^3-3(x+1) \\ &= (x+1)[(x+1)^2-3] \\ &= (x+1)(x^2+2x-2). \end{aligned}$$

Ex. 3. Factor x^3-3x+2.

Subtracting 1 from, and adding 1 to the given expression, we obtain

$$\begin{aligned} x^3-3x+2 &= x^3-1-3x+3 \\ &= (x^3-1)-3(x-1) \\ &= (x-1)[(x^2+x+1)-3] \\ &= (x-1)(x^2+x-2) \\ &= (x-1)(x-1)(x+2). \end{aligned}$$

20. Another device consists in separating a term into two or more terms, and grouping these component terms with others of the given expression.

Ex. Factor $x^3 - 3x^2 + 4$.

Separating $-3x^2$ into $-2x^2$ and $-x^2$, we obtain

$$\begin{aligned} x^3 - 3x^2 + 4 &= x^3 - 2x^2 - x^2 + 4 \\ &= x^2(x-2) - (x^2 - 4) \\ &= (x-2)[x^2 - (x+2)] \\ &= (x-2)(x^2 - x - 2) \\ &= (x-2)(x-2)(x+1) \\ &= (x-2)^2(x+1). \end{aligned}$$

21. These devices could have been applied to factor many expressions given under the preceding type-forms.

Ex. 1.
$$\begin{aligned} a^2 - b^2 &= a^2 - ab + ab - b^2 \\ &= a(a-b) + b(a-b) \\ &= (a+b)(a-b). \end{aligned}$$

Ex. 2.
$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 2x + 3x + 6 \\ &= x(x+2) + 3(x+2) \\ &= (x+2)(x+3). \end{aligned}$$

22. Symmetry. — An expression is *symmetrical* with respect to two letters if it remain the same when these letters are interchanged.

E.g., ab, $a^3 + b^3$, $a^2 + 2ab + b^2$ are symmetrical with respect to a and b, since, when a and b are interchanged, they become

ba, $b^3 + a^3$, $b^2 + 2ba + a^2$, as above.

23. An expression is symmetrical with respect to three or more letters, if it be symmetrical with respect to any two of them.

E.g., $a(b+c) + b(c+a) + c(a+b)$ is symmetrical with respect to the three letters a, b, and c. For, if any two letters, say a and c, be interchanged, we obtain

$c(b+a) + b(a+c) + a(c+b)$, as above.

24. Cyclo-symmetry. — An expression is cyclo-symmetrical with respect to three or more letters if it remain the same when the first letter is changed into the second, the second into the third, and so on, and the last into the first.

Such an interchange of letters is called a *cyclic interchange*.

Thus, abc becomes bca by a first cyclic interchange;
becomes cab by a second cyclic interchange;
becomes abc by a third cyclic interchange.

Therefore abc is a cyclo-symmetrical expression with respect to a, b, and c.

The expression $(a-d)(b^2-c^2)+(b-d)(c^2-a^2)+(c-d)(a^2-b^2)$, is cyclo-symmetrical with respect to a, b, and c. For, after making a cyclic interchange of these letters, we have

$$(b-d)(c^2-a^2)+(c-d)(a^2-b^2)+(a-d)(b^2-c^2), \text{ as above.}$$

Observe that this expression is *not* cyclo-symmetrical with respect to all four letters, a, b, c, and d.

25. A symmetrical or cyclo-symmetrical expression can frequently be factored by arranging its terms to powers of one of the letters with respect to which it is symmetrical.

Ex. Factor $\quad b^3(c-a)+c^3(a-b)+a^3(b-c)$.

Arranging the given expression to descending powers of a, we have

$$\begin{aligned}
&a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)\\
&\quad=(b-c)[a^3-a(b^2+bc+c^2)+bc(b+c)]\\
&\quad=(b-c)[a^3-ab^2-abc-ac^2+b^2c+bc^2]\\
&\quad=(b-c)[a(a^2-c^2)-b^2(a-c)-bc(a-c)]\\
&\quad=(b-c)(a-c)[a(a+c)-b^2-bc]\\
&\quad=(b-c)(a-c)[a^2+ac-b^2-bc]\\
&\quad=(b-c)(a-c)[(a^2-b^2)+c(a-b)]\\
&\quad=(b-c)(a-c)(a-b)(a+b+c).
\end{aligned}$$

EXERCISES IX.

Factor the following expressions:

1. $1+4x^4$.
2. $1+64x^4$.
3. $x^{4n}+4y^{4n}$.
4. $1+3a^2+4a^4$.
5. $1-7a^2+a^4$.
6. $1+2x^2y^2+9x^4y^4$.
7. $x^4-x^2y^2+16y^4$.
8. $x^4+y^4-11x^2y^2$.
9. $16x^4-x^2y^2+y^4$.
10. $x^4+4y^4-12x^2y^2$.
11. $x^4+y^8+x^2y^4$.
12. $x^8+y^8-142x^4y^4$.
13. x^3-6x^2+16.
14. x^3-15x^2+250.
15. $x^3+6x^2+10x+4$.
16. $x^3-9x^2+32x-42$.
17. $x^3-15x^2+72x-110$.
18. $8x^3-36x^2+48x-18$.
19. $27x^3-27x^2-6x+4$.
20. $ac(a-c)-ab(a-b)-bc(b-c)$.
21. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
22. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
23. $(a-d)(b^2-c^2)+(b-d)(c^2-a^2)+(c-d)(a^2-b^2)$.
24. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
25. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.

Method of Substitution.

26. In Ch. VI., § 2, Art. 4, it was proved that if an expression be divisible by $x-a$ (*i.e.*, if $x-a$ be a *factor* of the expression), the result of substituting a for x in the given expression is 0.

The following example will illustrate the method of applying this principle in factoring certain symmetrical expressions.

Ex. 1. Factor $(x-y)^3+(y-z)^3+(z-x)^3$.

The given expression has the factor $x-y$, if it be divisible by $x-y$; that is, if the result of substituting y for x be 0. Making this substitution, we have

$$0^3+(y-z)^3+(z-y)^3=(y-z)^3-(y-z)^3=0.$$

Therefore, $x-y$ is a factor of the given expression.

In like manner, it can be shown that $y-z$ and $z-x$ are factors. It is evident that the given expression, which is of the *third* degree, cannot have a fourth *literal* factor, since the product of four literal factors is an expression of the *fourth* degree. But it may be equal to the product of the factors $x-y$, $y-z$, $z-x$, and a numerical coefficient.

Let us assume, therefore,

$$(x-y)^3+(y-z)^3+(z-x)^3=C(x-y)(y-z)(z-x), \tag{1}$$

wherein C is some numerical coefficient yet to be determined.

Now (1) is an identity, and hence its first member must be equal to its second member for all values of the letters x, y, and z (Ch. IV., § 1, Art. 3).

If we substitute $x=0$, $y=1$, $z=2$, in both members of (1), we have

$$(-1)^3+(1-2)^3+2^3=C(-1)(1-2)\times 2.$$

Whence $$C=3.$$

Consequently, $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

This method depends upon the possibility of forming in advance of factoring a given expression some idea of the nature of its factors.

Ex. 2. Factor $(a+b-c)^2(a-b+c)+(a+b+c)(a+b-c)(b+c-a)$.

The given expression has the factor b if it be exactly divisible by $b-0$; that is, if the result of substituting 0 for b be 0. Making this substitution, we have

$$(a-c)^2(a+c)+(a+c)(a-c)(c-a)=(a-c)^2(a+c)-(a+c)(a-c)^2=0.$$

Therefore, b is a factor of the given expression. In like manner, it can be shown that c is a factor, and that a is not a factor.

It is evident that the given expression reduces to 0, when $a+b-c=0$. Therefore, $a+b-c$ is a factor. We now assume

$$(a+b-c)^2(a-b+c)+(a+b+c)(a+b-c)(b+c-a)=Cbc(a+b-c).$$

In this identity let $a = 0$, $b = 1$, $c = 2$. We thus obtain $C = 4$. Therefore,

$$(a+b-c)^2(a-b+c)+(a+b+c)(a+b-c)(b+c-a)=4\,bc\,(a+b-c).$$

In factoring expressions like the above, it is advisable to test such expressions as $a+b$, $a-b$, $a+b+c$, $a+b-c$, a, b, etc., according as any one of them is suggested by the form of the expression to be factored.

27. Ex. 1. Factor $x^3 - 5x^2 + 9x - 6$.

If this expression is a product of factors of the form $x - a$, then -6 is evidently the product of the last terms of these factors. Therefore, a must be a factor of 6, and possible values of a are 1, -1, 2, -2, 3, -3, 6, -6.

The given expression does not reduce to 0, when 1 and -1 are substituted in turn for x; therefore, $x - 1$ and $x - (-1)$, $= x + 1$ are not factors. But when $x = 2$, the given expression reduces to 0; therefore, $x - 2$ is a factor. The other factor is found by dividing the given expression by $x - 2$. We then have

$$x^3 - 5x^2 + 9x - 6 = (x-2)(x^2 - 3x + 3).$$

The factor $x^2 - 3x + 3$ is found not to have a factor of the form $x - a$. But in many examples this second expression can be factored.

Ex. 2. Factor $3x^3 - 5x^2 - 34x + 24$.

This expression reduces to 0 when $x = 4$. We then obtain

$$(x-4)(3x^2 + 7x - 6) = (x-4)(x+3)(3x-2).$$

EXERCISES X.

1–6. Factor the expressions given in Exercises IX., Exx. 20–25.

Factor the following expressions:

7. $a(b+c)^2 + b(c+a)^2 + c(a+b)^2 - 4abc$.

8. $(b+c-a)^2(a-b+c) + (a+b+c)(b+c-a)(a+b-c)$.

9. $(a+b)^2 + (a+c)^2 - (c+d)^2 - (b+d)^2$.

10. $(x+y+z)(xy+yz+zx) - xyz$.

11. $(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$.

12. $(x+y+z)^3 - (x+y-z)^3 - (x+z-y)^3 - (z+y-x)^3$.

13. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

14. $x^2y + y^2z + z^2x - xy^2 - yz^2 - x^2z$.

15. $(a+b+c)^3 - (a^3+b^3+c^3)$.

16. $(a+b+c)^4 + a^4 + b^4 + c^4 - (b+c)^4 - (c+a)^4 - (a+b)^4$.

17. $2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$.

18. $x^3 - 10x^2 + 31x - 30$. **19.** $x^3 + 10x^2 + 27x + 18$.

20. $5x^3 - 8x^2 - 27x + 18$. **21.** $x^3 - 5x^2 - 22x - 16$.

22. $x^3 - 21x + 20$. **23.** $x^3 - x^2 + 3x + 5$.

24. $3x^3 - 8x^2 - 10x + 21$. **25.** $x^3 - 6x^2 + 11x - 6$.

26. $x^3 + 18x^2 + 107x + 210$. **27.** $x^3 - 7x^2 - 21x + 27$.

28. $3x^3 - x^2 - 38x - 24$. **29.** $x^3 - 19x - 30$.

30. $3x^3 + 5x^2 - 7x - 1$. **31.** $2x^3 + 7x^2 + 4x + 3$.

EXERCISES XI.

Factor the following expressions by the methods given in this chapter:

1. $a^4 + 2a^3b - 2ab^3 - b^4$. **2.** $ax^2 + (a + b + c)x + b + c$.

3. $10c^{4n+1} - 5c^{7n+1} - 5c^{n+1}$. **4.** $x^2y^2 + 17xy + 16$.

5. $6a^4 - 6a^3c + 2a^3x^2 - 2a^2cx^2 + 6a^3x + 2a^2x^3$.

6. $a^6 - a^6z^4 + 3a^4z^2 - a^4z^6 + 3a^2z^4 + z^6$.

7. $x^6 + 64$. **8.** $a^6b^6 + 1$. **9.** $x^7 + y^7$. **10.** $a^{10} - a$.

11. $a^7 - 1$. **12.** $2^{3x+3} - 64$. **13.** $x^5y^5 - 1$. **14.** $2a^4 - 16ab^3$.

15. $x^4 + 2x^2 + 9$. **16.** $24x^2 - (3b - 8a)x - ab$.

17. $b^2 - c^2 + a(a - 2b)$. **18.** $x^{2m-2} + 2x^{m+n} + x^{2n+2}$.

19. $x^4 - 2x^3 - 1 + 2x$. **20.** $3x^2 + 4y^2 + z^2 - 8xy - 4yz + 4xz$.

21. $x^{3n+3} - 3x^{2n+2}y^2 + 3x^{n+1}y^4 - y^6$.

22. $ab(x^2 + y^2) + xy(a^2 + b^2)$.

23. $27x^3 + 27x^2y + 9xy^2 + y^3 - x^6 - 3x^4 - 3x^2 - 1$.

24. $28(x + 3)^2 - 23(x^2 - 9) - 15(x - 3)^2$.

25. $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$.

26. $(ax - by)^2 - (a + b)(x + z)(ax - by) + (a + b)^2xz$.

27. $ax^5 + bx^4 + cx^3 - ax^2 - bx - c$. **28.** $(a + b)x^2 + (a - 2b)x - 3b$.

29. $a^2 - b^2 - c^2 - 2a + 2bc + 1$. **30.** $49x^4y^6 + 42x^7y^9 + 9x^{10}y^{12}$.

31. $x^2 - 2x + 1 - y^2$. **32.** $15x^2 + x - 40$. **33.** $x^3 - x^2z + xz^2 - z^3$.

34. $a^3 - 1 + c - ac$. **35.** $a^2 - a - 1 - a^2c + ac + c$.

36. $2a^2 + a - 4ax - x + 2x^2$. **37.** $20x^2 - 123x + 180$.

38. $x^3 - 5x^2 - x + 5$. **39.** $x^2(x + 1) - b^2(b + 1)$.

40. $25a^4b^4 + 70a^2b^2c^2 + 49c^4$. **41.** $x^4y + zx^3 - xy - z$.

42. $x^2 - 9z^2 - 4y(y + 3z)$. **43.** $x^8 - 2x^4y^4 + y^8 - 4x^2y^2(x^2 - y^2)^2$.

44. $a^3 + a^2c + abc + b^2c - b^3$. **45.** $x^6 - 2x^5y + 2x^3y^3 - 2xy^5 + y^6$.

46. $3(a - 1)^3 - (1 - a)$. **47.** $x^6 - y^6 + 1 - 2x^3$.

48. $x^2 - ax - bx + ab$. **49.** $x^2y^2 + 25 - 9z^2 - 10xy$.

50. $x^2 + 9 - 2x(3 + 2xy^2)$. **51.** $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$

52. $3x^6 + 8x^4 - 8x^2 - 3$. **53.** $7a^3x^2 + 49a^2x + 84a$.

54. $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$. **55.** $cd - bd + a(b - c)$.

56. $8x^3 - 60x^2 + 140x - 100$. **57.** $(x^2 + 1)^3 - (y^2 + 1)^3$.

58. $abx^3 + x + ab + 1$. **59.** $36a^4 - 21a^2 + 1$.

60. $10x^4 - 47x^2 + 42$. **61.** $(x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2$.

62. $x^2+c(a+b)x+ab(a+c)(c-b)$. **63.** $5a^2 - 180b^2$.

64. $\frac{7}{18}abc^2 - \frac{7}{50}abd^2$. **65.** $10x^2 + 3x - 18$.

66. $x^{2n} - y^{2n} + 4y^n - 4x^n$. **67.** $ab(x^2 - y^2) + xy(a^2 - b^2)$.

68. $125x^3 - 150x^2 + 45x - 2$. **69.** $36a^4b^2 - 60a^3b^3 + 25a^2b^4$.

70. $a^2(a^2 - 1) - b^2(b^2 - 1)$. **71.** $(m - n)^2 - 12(m - n) + 27$.

72. $a^2x^5(a^3 - x) - a^5x^2(x^3 - a)$.

73. $3(a - 1)(a^5 + 7)^2 - 12(4a^5 + 28)(a - 1) + 192(a - 1)$.

74. $(x + y)^2 - 18(x + y) + 77$. **75.** $(a^2 - b^2)x^2 - (a^2 + b^2)x + ab$.

76. $300abc^2 - 432abd^2$. **77.** $75a^2b^2 - 108c^2d^2$.

78. $\frac{10}{25}abx^2y^2 - \frac{10}{49}abz^2$. **79.** $18a^2x^2 - 98b^2y^2$.

80. $18(x + y)^2 + 23(x^2 - y^2) - 6(x - y)^2$.

81. Express $(a^2 - b^2)(c^2 - d^2)$ as the difference of two squares.

§ 2. HIGHEST COMMON FACTORS.

1. If two or more integral algebraic expressions have no common factor except 1, they are said to be *prime to one another*, or are called *relatively prime expressions.*

E.g., ab and cd; $5x^2y$ and $8z^3$; $a^2 + b^2$ and $a^2 - b^2$.

2. The **Highest Common Factor (H. C. F.)** of two or more integral algebraic expressions is the expression of highest degree which exactly divides each of them.

E.g., the H. C. F. of ax^2, bx^3, and cx^4 is evidently x^2.

H. C. F. by Factoring.

3. Monomial Expressions.—The H. C. F. of monomials can be found by inspection.

Ex. 1. Find the H. C. F. of x^2y^5z, $x^4y^3z^2$, and $x^3y^4z^4$.

The expression of highest degree which exactly divides each of the given expressions evidently cannot contain a higher power of x than x^2, a higher power of y than y^3, and a higher power of z than z. Therefore the required H. C. F. is x^2y^3z.

Observe that the power of each letter in the H. C. F. is the *lowest* power to which it occurs in any of the given expressions.

If the monomials contain numerical factors, the Greatest Common Measure (G. C. M.) of these factors should be found as in Arithmetic.

Ex. 2. Find the H. C. F. of $18\,a^4b^5c^3d$, $42\,a^3bc^4$, and $30\,a^2b^2c^2$.

The G. C. M. of the numerical coefficients is 6. The lowest power of a in any of the given expressions is a^2; the lowest power of b is b; the lowest power of c is c^2; and d is not a common factor. Therefore the required H. C. F. is $6\,a^2bc^2$.

In general, *the H. C. F. of two or more monomials is obtained by multiplying the G. C. M. of their numerical coefficients by the product of their common literal factors, each to the lowest power to which it occurs in any of the given monomials.*

EXERCISES XII.

Find the H. C. F. of the following expressions:

1. ax^2, a^2x.

2. $15\,a$, $20\,a^2$, $10\,b$, 5.

3. a^2bx^2, ab^2x^2, a^2b^2x.

4. $2\,x^4$, $3\,x^5$, x^3, $5\,x^6$.

5. $56\,xy^3$, $70\,x^3y$, $98\,x^3y^3$.

6. $20\,a^2x^4b$, $40\,ax^3$, $10\,a^2x^3$.

7. $20\,a^4b^2$, $12\,a^3$, $10\,a^2b$.

8. $55\,x^2b^4$, $20\,x^2b^3$, $15\,a^2b^3x$, $5\,a^4b^4$.

9. $15\,a^2x^n$, $20\,a^3x^{n-1}$, $10\,ax^{n+1}$, $5\,a^2x^{n+2}$, $25\,a^3x^n$.

10. $(x-y)^2(x+z)^3$, $(x-y)^3(x+z)^2$.

4. Multinomial Expressions.—The method of finding the H. C. F. of multinomials by factoring is similar to that of finding the H. C. F. of monomials.

Ex. 1. The expressions

$$x^2-1=(x-1)(x+1),$$

and

$$x^2+x-2=(x-1)(x+2),$$

have only the common factor $x-1$. This is their H. C. F.

Ex. 2. The expressions

$$4x^4y^2 - 4x^2y^4 + 8x^3y^3 - 8xy^5 = 4xy^2(x^3 - xy^2 + 2x^2y - 2y^3)$$
$$= 4xy^2(x + 2y)(x + y)(x - y),$$
$$2x^4y - 2x^2y^3 + 2x^3y - 2xy^2 = 2xy(x^3 - xy^2 + x^2 - y^2)$$
$$= 2xy(x + 1)(x + y)(x - y),$$

have the common factors $2xy$, $x + y$, $x - y$.

Therefore their H. C. F. is

$$2xy(x + y)(x - y) = 2xy(x^2 - y^2).$$

In general, *the H. C. F. of two or more multinomial expressions is the product of their common factors, each to the lowest power to which it occurs in any of them.*

EXERCISES XIII.

Find the H. C. F. of the following expressions :

1. $a^3 + b^3,\ a^2 - b^2.$
2. $8 - a^3,\ a^2 - 4.$
3. $x^4 - y^4,\ (x^2 - y^2)^2.$
4. $x^5 - y^5,\ ax - ay.$
5. $x^2 + xy,\ x^5 + y^5.$
6. $x^5 - y^5,\ x^2 - y^2.$
7. $x^2 + xy,\ x^2 + 2xy + y^2.$
8. $a^2 - ab,\ a^2 - 2ab + b^2.$
9. $x^2 - 1,\ x^2 + 2x + 1.$
10. $x^3 - 1,\ x^2 + x + 1.$
11. $x^3 + 27,\ x^2 + 6x + 9.$
12. $x^3 + 1,\ x^3 + mx^2 + mx + 1.$
13. $x^2 + 2xy + y^2 - a^2,\ 2x + 2y + 2a.$
14. $x^4 + 9x^2 + 20,\ x^4 + 7x^2 + 10.$
15. $3x^3 - 8x^2 + 4x,\ x^3 - 6x^2 + 12x - 8.$
16. $3x^2 - ax - 4a^2,\ 6x^2 - 17ax + 12a^2.$
17. $a^3 + 2a^2 + 2a + 1,\ a^3 + 1.$
18. $x^3 - 1,\ x^2 - 1,\ (x - 1)^2.$
19. $x^2 + 5x + 4,\ x^2 + 2x - 8,\ x^2 + 7x + 12.$
20. $x^2 - 2a^2 - ax,\ x^2 - 4a^2,\ x^2 - 6a^2 + ax.$
21. $x^2 - 2x - 3,\ x^2 - 7x + 12,\ x^2 - x - 6.$
22. $x^6 - y^6,\ x^4 + xy^3,\ x^6 + 2x^3y^3 + y^6.$

H. C. F. by Division.

5. A **Multiple** of an integral algebraic expression is an expression which is exactly divisible by the given one.

E.g., multiples of $a + b$ are $2(a + b)$, $(x - y)(a + b)$, etc.

6. If the given expressions cannot be readily factored, their H. C. F. can be obtained by a method analogous to that used in Arithmetic to find the G. C. M. of numbers.

7. The expressions whose H. C. F. is required should be arranged to powers of a common letter of arrangement.

If one of two expressions be divisible without a remainder by the other, which must be of the same or lower degree in the letter of arrangement, then the latter (the divisor) is the required H. C. F. For it is a factor of the other expression.

But if the one expression be not divisible without a remainder by the other, their H. C. F. is found by applying the following principles:

(i.) *A common factor of two integral algebraic expressions is also a factor of the sum or the difference of any multiples of the expressions (including simply the sum or the difference of the expressions).*

E.g., $x-y$ is a common factor of x^2-y^2 and $x^2-2xy+y^2$;

also of $(x^2-y^2)-(x^2-2xy+y^2)=-2y^2+2xy=2y(x-y)$,

and $3(x^2-y^2)+2(x^2-2xy+y^2)=5x^2-4xy-y^2=(5x+y)(x-y)$.

(ii.) *If an integral algebraic expression be divided by another (of the same or lower degree in a common letter of arrangement) and if there be a remainder, then the H. C. F. of this remainder and the divisor is the H. C. F. of the given expressions.*

E.g., the H. C. F. of

$$x^4-10x^3+35x^2-50x+24, =(x-1)(x-2)(x-3)(x-4), \quad (1)$$

and

$$x^3-7x^2+11x-5, =(x-1)(x-1)(x-5) \quad (2)$$

is $x-1$. The remainder obtained by dividing (1) by (2) is

$$3x^2-12x+9, =3(x-1)(x-3). \quad (3)$$

The H. C. F. of this remainder and the divisor (2) is evidently also $x-1$, the H. C. F. of (1) and (2).

Notice that the H. C. F. of the remainder and the dividend (1) is $(x-1)(x-3)$, and is *not* the H. C. F. of (1) and (2).

(i.) Let E_1 and E_2 stand for the two expressions, and F for their common factor. Then we are to prove that F is a factor of

$$ME_1 + NE_2,$$

wherein M and N stand for two numbers or integral algebraic expressions.

Since F is a factor of E_1, E_1 is the product of F and some other expression, say Q_1; or

$$E_1 = Q_1F. \qquad (1)$$

For a similar reason $E_2 = Q_2F$. (2)

Multiplying both members of (1) by M, and both members of (2) by N, we have, by Ch. I., § 1, Art. 15 (iii.),

$$ME_1 = MQ_1F, \text{ and } NE_2 = NQ_2F.$$

Adding corresponding members of the last equations, we have, by Ch. I., § 1, Art. 15 (i.),

$$ME_1 + NE_2 = MQ_1F + NQ_2F = (MQ_1 + NQ_2)F.$$

The last equation shows that $ME_1 + NE_2$ contains F as a factor, the remaining factor being $MQ_1 + NQ_2$.

(ii.) Let E_1 and E_2 stand for two integral algebraic expressions, which have a common factor, and let E_1 be of the same or higher degree than E_2 in some letter of arrangement.

Let Q be the quotient and R the remainder of dividing E_1 by E_2. Then, by Ch. III., § 4, Art. 13, we have

$$E_1 = QE_2 + R. \qquad (1)$$

By (i.), any common factor of R and E_2, *and therefore their H. C. F.*, is also a factor of $QE_2 + R$; that is, of E_1. Transferring QE_2 to the first member of (1), we obtain

$$E_1 - QE_2 = R. \qquad (2)$$

From the last equation we infer that any common factor of E_1 and E_2, *and therefore their H. C. F.*, is also a factor of $E_1 - QE_2$; that is, of R.

If now the H. C. F. of E_1 and E_2 be not the H. C. F. of R and E_2, then E_2 must have in common with R some factor of higher degree than is contained in E_1. But this contradicts the first part of the proof, that the H. C. F. of R and E_2 is also a factor of E_1.

8. The following example will illustrate the method of applying the principles of Art. 7:

Ex. Find the H. C. F. of $x^2 - 3x + 2$, (1)

and $x^3 - 4x^2 + 4x - 1$. (2)

Dividing (2) by (1), we have

$$\begin{array}{r|l}
x^3-4x^2+4x-1 & x^2-3x+2 \\ \cline{2-2}
\underline{x^3-3x^2+2x} & x-1 \\
-x^2+2x & \\
\underline{-x^2+3x-2} & \\
-x+1 &
\end{array}$$

By Art. 7 (ii.), the H. C. F. of (1) and (2) is the H. C. F. of

x^2-3x+2 and the remainder $-x+1$.

We change the sign of this remainder, since, if x^2-3x+2 is divisible by $-x+1$, it is divisible by $x-1$, $=-(-x+1)$.

We now have
$$\begin{array}{r|l}
x^2-3x+2 & x-1 \\ \cline{2-2}
\underline{x^2-x} & x-2 \\
-2x+2 & \\
\underline{-2x+2} &
\end{array}$$

Since the remainder of this division is 0, the divisor $x-1$ (*i.e.*, the remainder of the first division) is the H. C. F. of itself and (1), and therefore of (1) and (2).

This work can be arranged more compactly thus:

$$\begin{array}{rl}
x^2-3x+2) & x^3-4x^2+4x-1(x-1 \\
& \underline{x^3-3x^2+2x} \\
& -x^2+2x \\
& \underline{-x^2+3x-2} \\
\times(-1) & \underline{-x+1|} \\
& x-1)\,x^2-3x+2\,(x-2 \\
& \underline{x^2-x} \\
& -2x \\
& \underline{-2x+2}
\end{array}$$

The divisor is, for convenience, placed on the left of the dividend, and the quotient on the right.

9. The following principle will frequently simplify the work of finding the H. C. F. of two expressions:

Either of the expressions may be multiplied or divided by any number which is not already a factor of the other expression.

For a factor introduced by multiplication into one expression will not be common to both of them, and therefore will not be introduced into their H. C. F.

In like manner, the factor removed by division from one expression was not common to both of them, and therefore would not have been a factor of their H. C. F.

Ex. 1. Find the H. C. F. of

$$\begin{aligned}&2x^5y^2-12x^4y^2+12x^3y^2-6x^2y^2+4xy^2\\&\quad=2xy^2(x^4-6x^3+6x^2-3x+2),\\&6x^5y-15x^4y+21x^3y-12x^2y\\&\quad=3x^2y(2x^3-5x^2+7x-4).\end{aligned}$$

We set aside xy, the H. C. F. of $2xy^2$ and $3x^2y$, as a factor of the required H. C. F., and find the H. C. F. of the remaining factors by division.

The first of these expressions cannot be divided by the second without introducing fractional coefficients. To avoid these we multiply the first by 2, *since 2 is not a factor of the other expression.*

$$\begin{array}{rl}
2x^3-5x^2+7x-4) & 2x^4-12x^3+12x^2-6x+4(x+7\\
 & \underline{2x^4-5x^3+7x^2-4x}\\
\times(-2) & \underline{-7x^3+5x^2-2x+4}\\
 & 14x^3-10x^2+4x-8\\
 & \underline{14x^3-35x^2+49x-28}\\
\div 5 & \underline{25x^2-45x+20}
\end{array}$$

2d divisor,

$$\begin{array}{rl}
5x^2-9x+4) & 10x^3-25x^2+35x-20(2x-7\\
 & \underline{10x^3-18x^2+8x}\\
\times 5 & \underline{-7x^2+27x-20}\\
 & -35x^2+135x-100\\
 & \underline{-35x^2+63x-28}\\
\div 72 & \underline{72x-72}
\end{array}$$

3d divisor,

$$\begin{array}{rl}
x-1) & 5x^2-9x+4(5x\\
 & \underline{5x^2-5x}\\
 & -4x\\
 & \underline{-4x+4}
\end{array}$$

To avoid fractional coefficients, we multiply the partial remainder of the first division by -2, divide the remainder of the first division by 5. In beginning the second stage of the work, the dividend is the first divisor multiplied by 5. To avoid fractional coefficients, we multiply the partial remainder of the second division by 5, and divide the remainder of the second division by 72. The required H. C. F. is $xy(x-1)$.

Ex. 2. Find the H. C. F. of

$$x^4-10x^3+35x^2-50x+24 \qquad (1)$$

and

$$x^3-7x^2+11x-5. \qquad (2)$$

We have:

$$\begin{array}{lr}
\text{1st divisor, } x^3-7x^2+11x-5) & x^4-10x^3+35x^2-50x+24(x-3 \\
 & \underline{x^4-7x^3+11x^2-5x} \\
 & -3x^3+24x^2-45x \\
 & \underline{-3x^3+21x^2-33x+15} \\
 & \div\ 3 \qquad \underline{\left\lfloor 3x^2-12x+9\right.} \\
 & x^2-4x+3
\end{array}$$

The remainder $x^2-4x+3, =(x-1)(x-3)$, is readily factored.

Dividing $x^3-7x^3+11x-5$ by $x-1$, we have

$$x^3-7x^2+11x-5=(x-1)(x^2-6x+5)=(x-1)^2(x-5).$$

The H. C. F. of the first remainder and (2), and therefore the required H. C. F., is $x-1$.

10. The examples worked in the preceding articles illustrate the following method of finding the H. C. F. of two expressions:

(i.) *Remove from the given expressions any monomial factors, and set aside their H. C. F. as a factor of the required H. C. F.*

(ii.) *Divide the expression of higher degree in a common letter of arrangement by the one of lower degree; if the expressions be of the same degree, either may be taken as the first divisor.*

(iii.) *Divide the first divisor by the first remainder, the first remainder (second divisor) by the second remainder, and so on, until a remainder* 0 *is obtained. The last divisor will be the required H. C. F.*

If a remainder which does not contain the letter of arrangement, and which is not 0, is obtained, the given expressions do not have a H. C. F. in this letter of arrangement.

(iv.) *At any stage of the work the dividend may be multiplied by any number which is not a factor of the corresponding divisor; or the divisor may be divided by any number which is not a factor of the corresponding dividend.*

(v.) *If the divisor and dividend at any stage of the work can be factored readily, it is better to find their H. C. F. by factoring than by continuing the method of division.*

11. To find the H. C. F. of three or more integral algebraic expressions find the H. C. F. of any two of them, next the H. C. F. of that H. C. F. and the third expression, and so on.

For any common factor of three or more expressions, and hence their H. C. F., must be a factor of the H. C. F. of any two of them.

EXERCISES XIV.

Find the H. C. F. of the following expressions:

1. $x^3 + 4x - 5$, $x^3 - 2x^2 + 6x - 5$.
2. $2x^3 + 3x^2 - x - 12$, $6x^3 - 17x^2 + 2x + 15$.
3. $x^3 - 3x^2 + 4$, $x^3 - 2x^2 - 4x + 8$.
4. $x^2 - 3x + 2$, $x^4 - 6x^2 + 8x - 3$.
5. $2x^2 + 3x - 2$, $4x^3 + 16x^2 - 19x + 5$.
6. $x^3 - 3x^2 + 4$, $3x^3 - 18x^2 + 36x - 24$.
7. $x^3 - (a + b - c)x^2 + (ab - ac - bc)x + abc$,
   $x^3 - (a - b + c)x^2 + (ac - ab - bc)x + abc$.
8. $x^3 + x^2 - 5x + 3$, $2x^3 + 7x^2 - 9$.
9. $3x^3 - 8x^2 - 36x + 5$, $9x^3 - 50x^2 + 27x - 10$.
10. $4x^3y^3 - 3x^2y^2 - 4xy + 3$, $5x^3y^3 + 8x^2y^2 + xy - 14$.
11. $x^3 - 3xy^2 - 2y^3$, $2x^3 - 5x^2y - xy^2 + 6y^3$.
12. $a^3 - a^2 - 5a + 2$, $3a^3 - a^2 - 8a + 12$.
13. $14x^3 - 41x^2y + 17xy^2 - 5y^3$, $10x^3 - 31x^2y + 23xy^2 - 20y^3$.
14. $x^3 + 2x^2 + 2x + 1$, $x^3 - 4x^2 - 4x - 5$.
15. $30x^3 - 25ax^2 + 8a^2x - a^3$, $18x^3 - 24ax^2 + 15a^2x - 3a^3$.

16. $36a^6 + 9a^3 - 27a^4 - 18a^5,\ 27a^5b^2 - 9a^3b^2 - 18a^4b^2.$

17. $3x^5 - 10x^3 + 15x + 8,\ x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$

18. $2x^3 - 3x^2 - 8x - 3,\ 2x^4 - 9x^3 + 13x^2 - 23x - 16.$

19. $2a^3x^3 - 7a^2x^2 + 11ax - 15,\ 2a^4x^4 - 7a^3x^3 + 8a^2x^2 - 12ax - 9.$

20. $x^5 + x^3 - 8x^2 - 8,\ x^4 - 2x^3 + x^2 - 2x.$

21. $x^3 - 4x + 3,\ 2x^3 + x^2 - 7x + 4,\ x^3 - 2x^2 + 1.$

22. $2x^3 + 5x^2 - 4x - 10,\ 2x^3 + 5x^2 + 2x + 5,\ 2x^3 + 7x^2 + 7x + 5.$

23. $2x^4 + 6x^3 + 4x^2,\ 3x^3 + 9x^2 + 9x + 6,\ 3x^3 + 8x^2 + 5x + 2.$

24. $x^3 - 3x^2 - 4x + 12,\ x^3 - 7x^2 + 16x - 12,\ 2x^3 - 9x^2 + 7x + 6.$

25. $2x^4 - x^3 + 3x^2 + x + 4,\ 2x^4 - 3x^3 - 2x^2 + 9x - 12,$
$4x^4 - 16x^3 + 25x^2 - 23x + 4.$

12. The words *Highest Common Factor* in Algebra refer to the degree of the common factor. Thus, the H. C. F. of

$$x^3 - 2x^2 - x + 2 = (x^2 - 1)(x - 2),$$

and

$$x^3 - 4x^2 - x + 4 = (x^2 - 1)(x - 4)$$

is evidently $x^2 - 1$.

That factor is of higher degree in x than any other common factor, as $x - 1$, $x + 1$.

The words *Greatest Common Measure* refer to the greatest numerical common measure when particular numerical values are substituted for the letters.

If we substitute 6 for x in the above expressions, we have

$$x^3 - 2x^2 - x + 2 = (x^2 - 1)(x - 2) = 35 \times 4 = 140,$$

and

$$x^3 - 4x^2 - x + 4 = (x^2 - 1)(x - 4) = 35 \times 2 = 70.$$

The arithmetical G. C. M. of 70 and 140 is evidently 70.

Now notice that when $x = 6$, the G. C. M. of the expressions is not the same in numerical value as the H. C. F.; for when $x = 6$,

$$x^2 - 1 = 35, \text{ not } 70.$$

The reason for this is that while $x - 2$ and $x - 4$ do not have an algebraic common factor, their numerical values for particular values of x may have a common numerical factor.

Thus, when $x = 6$, $x - 2$ and $x - 4$ have the values 4 and 2, respectively, and therefore have the common factor 2.

The words *Greatest Common Measure* should not therefore be used in the same sense as the words *Highest Common Factor*.

13. The following principles will be of use in subsequent work:

(i.) *In the process for finding the arithmetical G. C. M. of two integers, M and N, the remainder at any stage of the work can be expressed in the form*

$$\pm(mM - nN),$$

wherein m and n are positive integers, and the upper sign goes with the first, third, etc., remainders, and the lower sign with the second, fourth, etc.

Let M be greater than N. Then, in the process for finding the G. C. M., let Q_1 be the quotient and R_1 the remainder of the first division, Q_2 and R_2 the quotient and the remainder, respectively, of the second division, and so on. It is to be kept in mind that the Q's and the R's are positive integers.

Then, by Ch. III., § 4, Art. 13, we have

$$M = Q_1N + R_1 \quad (1), \qquad N = Q_2R_1 + R_2 \quad (2) \qquad R_1 = Q_3R_2 + R_3 \quad (3), \text{ etc.}$$

Then, from (1): $R_1 = M - Q_1N;$ (4)

from (2): $R_2 = -Q_2R_1 + N$

$= -Q_2(M - Q_1N) + N,$

substituting the value of R_1 from (4)

$= -[Q_2M - (Q_1Q_2 + 1)N];$ (5)

from (3): $R_3 = -Q_3R_2 + R_1$

$= Q_3[Q_2M - (Q_1Q_2 + 1)N] + M - Q_1N$

$= (Q_2Q_3 + 1)M - (Q_1Q_2Q_3 + Q_1 + Q_3)N.$ (6)

In like manner, the value of each succeeding remainder, in terms of M and N, can be derived.

In (4), $m = 1$, $n = Q_1$; in (5) $m = Q_2$, $n = Q_1Q_2 + 1$, and so on.

(ii.) *If M and N be two positive integers, prime to each other, then two positive integers, m and n, can be found, such that*

$$mM - nN = \pm 1.$$

Since M and N are prime to each other, 1 is their G. C. M. Therefore, the next to the last remainder will be 1 (the last being 0). Consequently, by (i.) two positive integers, m and n, can be found such that

$$\pm(mM - nN) = 1; \text{ or } mM - nN = \pm 1.$$

(iii.) *If M and N be two positive integers, prime to each other, then any common factor of M and NR must be a factor of R.*

For by (ii.), $mM - nN = \pm 1.$

Therefore, $mMR - nNR = \pm R$, or $mR \cdot M - n \cdot NR = \pm R.$

Since, by Art. 7 (i.), any common factor of M and NR is a factor of $mR \cdot M - n \cdot NR$, the last equation shows that this factor is a factor of R.

The following principles follow directly from (iii.).

(iv.) *If M be a factor of NR and be prime to N, it is a factor of R.*

(v.) *If M be prime to R, S, etc., it is prime to $RS \cdots$.*

(vi.) *If each of the integers M, N, P be prime to each of the integers R, S, T, then MNP is prime to RST.*

(vii.) *If M be prime to N, then M^p is prime to N^p, wherein p is a positive integer.*

§ 3. LOWEST COMMON MULTIPLES.

1. The **Lowest Common Multiple (L. C. M.)** of two or more integral algebraic expressions is the integral expression of lowest degree which is exactly divisible by each of them.

E.g., the L. C. M. of ax^2, bx^3, and cx^4 is evidently $abcx^4$.

L. C. M. by Factoring.

2. Ex. 1. Find the L. C. M. of a^3b, a^2bc^2, and ab^2c^4.

The expression of lowest degree which is exactly divisible by each of the given expressions cannot contain a lower power of a than a^3, a lower power of b than b^2, and a lower power of c than c^4. Therefore, the required L. C. M. is $a^3b^2c^4$.

Observe that the power of each letter in the L. C. M. is the *highest* power to which it occurs in any of the given expressions. If the expressions contain numerical factors, the L. C. M. of these factors should be found as in Arithmetic.

Ex. 2. Find the L. C. M. of

$$3\,ab^2,\ 6\,b(x+y)^2, \text{ and } 4\,a^2b(x-y)(x+y).$$

The L. C. M. of the numerical coefficients is 12.

The highest power of a in any of the expressions is a^2; of b is b^2; of $x+y$ is $(x+y)^2$; and of $x-y$ is $x-y$.

Consequently the required L. C. M. is $12\,a^2b^2(x+y)^2(x-y)$.

In general, *the L. C. M. of two or more expressions is obtained by multiplying the L. C. M. of their numerical coefficients by the product of all the different prime factors of the expressions, each to the highest power to which it occurs in any of them.*

EXERCISES XV.

Find the L. C. M. of the following expressions:

1. $2a,\ 3b.$
2. $4a^3b,\ 2ab^2,\ 3ax.$
3. $4ab,\ 2a^2b^2,\ 12a^3b.$
4. $14a^3,\ 21a^2,\ 5b,\ 7a.$
5. $7a^2b,\ 3a^3bx,\ 2ab,\ 2a^2x^3.$
6. $7a^3m^2,\ 21x^2m^3,\ 343xm.$
7. $12a^3b^2x,\ 18x^3a^2b,\ 36ab^3x.$
8. $8a^2x^3,\ 30a^3x^3,\ 4a^2x^4,\ 10ax.$
9. $20a^2x^n,\ 15a^3x^{n-1},\ 10ax^{2n+1}.$
10. $5x+11,\ 10x-33.$
11. $a+1,\ a-1,\ a.$
12. $a+b,\ a^2+2ab+b^2.$
13. $x+1,\ x^2-2x-3.$
14. $a+x,\ a^2-x^2.$
15. $x-5,\ x^2-3x-10.$
16. $3x-3,\ a^2-2ax+x^2.$
17. $8a^2+16a,\ a^3+4a^2+4a,\ a^3.$
18. $a^2-b^2,\ 4a+4b,\ a^3-b^3-3a^2b+3ab^2.$
19. $x+1,\ x^2-1,\ x^3-1.$
20. $a^3-x^3,\ a^2-x^2,\ x-a.$
21. $x^2-y^2,\ (x-y)^2,\ x^3-y^3.$
22. $x-a,\ a^2-x^2,\ x^4-a^4.$
23. $1-2x,\ 4x^2-1,\ 1+4x^2.$
24. $1-x,\ x^2-1,\ x-2,\ x^2-4.$
25. $4(1-x)^2,\ 8(1-x),\ 8(1+x),\ 4(1-x^2).$
26. $9a^4b^2-4c^2d^4,\ 9a^4b^2-12a^2bcd^2+4c^2d^4.$
27. $3x^2-5x+2,\ 4x^3-4x^2-x+1.$
28. $x^2-4a^2,\ x^3+2ax^2+4a^2x+8a^3,\ x^3-2ax^2+4a^2x-8a^3.$
29. $6(a^3-b^3)(a-b)^3,\ 9(a^4-b^4)(a-b)^2,\ 12(a^2-b^2).$

Lowest Common Multiple by Means of H. C. F.

3. If the given expressions cannot be readily factored, their L. C. M. can be obtained by first finding their H. C. F.

Ex. Find the L. C. M. of

$$x^3-2x^2-2x^2y+4xy+x-2y \text{ and } x^3-2x^2y+xy^2-2y^3.$$

The H. C. F. of these expressions is found to be $x-2y$.

Consequently the other factors of the given expressions can be found by dividing each of them by their H. C. F. We have

$$x^3-2x^2-2x^2y+4xy+x-2y=(x-2y)(x^2-2x+1),$$

$$x^3-2x^2y+xy^2-2y^3=(x-2y)(x^2+y^2).$$

From the definition of the H. C. F. we know that these second factors, x^2-2x+1 and x^2+y^2, have no common factor, and therefore that the L. C. M. of the given expressions must contain both of them as factors.

Consequently the required L. C. M. is

$$(x-2y)(x^2+y^2)(x-1)^2.$$

4. The example of Art. 3 illustrates the following principle:

The L. C. M. of two integral algebraic expressions is the product of their H. C. F. by the remaining factors of the expressions.

Let E_1 and E_2 stand for the two expressions, and let F stand for their H. C. F.

Then $$E_1 = FQ_1 \text{ and } E_2 = FQ_2,$$

wherein Q_1 and Q_2 are the remaining factors of E_1 and E_2, respectively.

Since Q_1 and Q_2 have no common factor, the expression of lowest degree which is exactly divisible by FQ_1 and FQ_2 must be FQ_1Q_2.

5. To find the L. C. M. of three or more integral algebraic expressions, find the L. C. M. of any two of them; next, the L. C. M. of a third and the L. C. M. already found, and so on.

Let E_1, E_2, E_3 stand for three integral algebraic expressions, and let M_1 stand for the L. C. M. of E_1 and E_2, and M_2 for the L. C. M. of M_1 and E_3. Then M_2 is the expression of lowest degree which is exactly divisible by M_1 and E_3; but M_1 is the expression of lowest degree which is exactly divisible by E_1 and E_2. Hence M_2 is the expression of lowest degree which is exactly divisible by E_1, E_2, E_3.

EXERCISES XVI.

Find the H. C. F. and L. C. M. of the following expressions:

1. $x^3-x,\ x^3-1.$
2. $x^2-1,\ x^3+1.$
3. $x^3-3x+2,\ x^3+2x^2-x-2.$
4. $x^2+x-2,\ x^3+2x^2+2x+1.$
5. $2x^3-17x^2+19x-4,\ 3x^3-20x^2-10x+27.$
6. $6x^3-x^2+11x+4,\ 3x^3+13x^2+x-3.$
7. $x^3-5x^2+9x-9,\ x^4-4x^2+12x-9.$
8. $x^3-x^2-9x+9,\ x^4-4x^2+12x-9.$
9. $14x^3-17x^2+11x-3,\ 6x^4-3x^3+4x^2-1.$
10. $2x^4-3x^3+4x^2-5x-4,\ 2x^4-x^3+x-12.$
11. $4x^3-8x^2+5x-3,\ 2x^4-3x^3+6x^2-3x+2.$
12. $4x^4-8x^3-3x^2+7x-2,\ 3x^3-11x^2+2x+16.$
13. $x^3-6x^2+11x-6,\ x^3-9x^2+26x-24,\ x^3-8x^2+19x-12.$
14. $x^3-5x^2+9x-9,\ x^3-x^2-9x+9,\ x^4-4x^2+12x-9.$

Relation between H. C. F. and L. C. M.

6. The following example illustrates an important relation between the H. C. F. and the L. C. M. of two integral algebraic expressions.

Ex. The H. C. F. of

$$x^3 - 1 = (x-1)(x^2+x+1)$$

and

$$x^2 - 1 = (x-1)(x+1)$$

is

$$(x-1).$$

The L. C. M. of the same expressions is

$$(x-1)(x+1)(x^2+x+1).$$

The product of the two given expressions is

$$(x-1)(x-1)(x+1)(x^2+x+1) = (\text{H. C. F.}) \times (\text{L. C. M.}).$$

This example illustrates the principle:

The product of two integral algebraic expressions is equal to the product of their H. C. F. and their L. C. M.

Let E_1 and E_2 stand for two integral algebraic expressions, and let F stand for their H. C. F. and M for their L. C. M.

Then $$E_1 = Q_1F \text{ and } E_2 = Q_2F,$$

wherein, as before, Q_1 and Q_2 stand for the remaining factors of E_1 and E_2, respectively. The L. C. M. of the two expressions is Q_1Q_2F.

But the product of the two expressions is

$$Q_1FQ_2F = (Q_1Q_2F)F = M \cdot F.$$

It follows from this principle that the L. C. M. of two integral algebraic expressions can be found by dividing their product by their H. C. F.

§ 4. SOLUTION OF EQUATIONS BY FACTORING.

1. The roots of the equation

$$(x-1)(x-2) = 0 \qquad (1)$$

are evidently 1 and 2. For 1 reduces the first member to $0 \times (-1), = 0$; and 2 reduces the first member to $1 \times 0, = 0$. Therefore equation (1) is equivalent to the equations

$$x - 1 = 0 \text{ and } x - 2 = 0, \text{ jointly.}$$

This example illustrates the following principle:

If all the terms of an integral equation be transferred to the first member, and if this first member be factored, the given equation is equivalent to the set of equations obtained by equating to 0 each factor of its first member.

Let
$$P \times Q \times R = 0 \qquad (1)$$
be the given equation. Then we are to prove that the equation is equivalent to the set of equations
$$P = 0, \quad Q = 0, \quad R = 0. \qquad (2)$$
For any solution of (1) must reduce $P \times Q \times R$ to 0, and, therefore, by Ch. III., § 3, Art. 18, either P, or Q, or R to 0. That is, every solution of (1) is a solution of one of equations (2).

Any solution of $P = 0$ must reduce P to 0, and, therefore, by Ch. III., § 3, Art. 16, $P \times Q \times R$ to 0.

That is, every solution of $P = 0$ is a solution of $P \times Q \times R = 0$.

In like manner, it can be shown that the solutions of $Q = 0$ and $R = 0$ are solutions of (1).

Ex. Solve the equation $x(x-2)(x+5) = 0$.

The given equation is equivalent to the equations
$$x = 0, \quad x - 2 = 0, \quad \text{and} \quad x + 5 = 0.$$
The roots are therefore 0, 2, and -5.

EXERCISES XVII.

Solve the following equations:

1. $x(x-1) = 0$.
2. $5y(y+11) = 0$.
3. $(x+2)(2x-3) = 0$.
4. $(5x+4)(9-3x) = 0$.
5. $x(x-5)(3-2x) = 0$.
6. $5x(6x-7)(2-4x) = 0$.
7. $x^2 - 5x + 6 = 0$.
8. $10x^2 + 7x - 12 = 0$.
9. $x^3 + 6x^2 - 16x = 0$.
10. $x^3 - 3x^2 - 10x = 0$.
11. $(x^2-4)(x^2-9) = 0$.
12. $(9x^2-25)(12-5x-2x^2) = 0$.

2. The expression $(x-1)(x-2)$ reduces to 0 for $x = 1$ and $x = 2$, and the expression $(x-5)(x+4)$ reduces to 0 for $x = 5$ and $x = -4$.

Observe that the two expressions do not have a common factor, and do not reduce to 0 for the same values of x. This example illustrates the following principle:

If two expressions in one and the same unknown number do not have a common factor, they cannot reduce to 0 for the same value of the unknown number.

Let E_1 and E_2 be two integral expressions in x which do not have a common factor. Then we are to prove that E_1 and E_2 cannot reduce to 0 for the same value of x.

For if E_1 and E_2 do reduce to 0 for the same value of x, say a, they must both be divisible by $x - a$ without a remainder (Ch. VI., § 2, Art. 4). That is, $x - a$ must be a factor of both expressions. But this contradicts the hypothesis that E_1 and E_2 do not have a common factor.

CHAPTER IX.

FRACTIONS.

1. The quotient of a division can be expressed as an integer or an integral expression only when the dividend is a multiple of the divisor; as $a^2b \div ab = a$; $(ax^2 + 2\,bx) \div x = ax + 2\,b$.

If the dividend be not a multiple of the divisor, the quotient is called a **Fraction**; as $a \div b$; $(ax^2 + 2\,bx) \div x^3$.

2. The notation for a fraction in Algebra is the same as in ordinary Arithmetic.

Thus, $(ax^2 + 2\,bx) \div x^3$ is written $\dfrac{ax^2 + 2\,bx}{x^3}$.

The **Solidus**, /, is frequently used instead of the horizontal line to denote a fraction; as $(ax^2 + bx)/x^3$ for $\dfrac{ax^2 + bx}{x^3}$.

3. As in Arithmetic, the dividend is called the **Numerator** of the fraction, the divisor the **Denominator**, and the two are called the **Terms** of the fraction.

4. An integer or an integral expression can be written in a *fractional form* with a denominator 1.

E.g., $7 = \dfrac{7}{1}$, $a + b = \dfrac{a+b}{1}$.

It is important to notice that an algebraic fraction may be *arithmetically* integral for certain values of its terms.

E.g., when $a = 4$ and $b = 2$, the fraction a/b becomes $4/2 = 2$.

5. By the definition of a fraction, a/b is a number which, multiplied by b, becomes a; that is,

$$(a/b) \times b = a, \text{ or } \frac{a}{b} \times b = a \qquad (1)$$

6. The Sign of a Fraction. — The sign of a fraction is written before the line separating its numerator from its denominator; as $+\frac{a}{b}$, $-\frac{a}{b}$.

Since a fraction is a quotient, the sign of a fraction is determined by the rule of signs in division.

$$\frac{+a}{+b}=+\frac{a}{b},\quad \frac{-a}{-b}=+\frac{a}{b},\quad \frac{+a}{-b}=-\frac{a}{b},\quad \frac{-a}{+b}=-\frac{a}{b}.$$

7. From the rule of signs we derive:

(i.) *If the signs of the numerator and the denominator of a fraction be reversed, the sign of the fraction is unchanged.*

E.g., $$\frac{-7}{3}=\frac{7}{-3};\quad \frac{x}{x-1}=\frac{-x}{1-x}.$$

This step is equivalent to multiplying or dividing both terms of the fraction by -1.

(ii.) *If the sign of either the numerator or the denominator of a fraction be reversed, the sign of the fraction is reversed; and conversely.*

E.g., $$\frac{7}{3}=-\frac{-7}{3};\quad \frac{-x}{x-1}=-\frac{x}{x-1};\quad -\frac{x-a}{b-x}=\frac{x-a}{x-b}.$$

(iii.) *If the signs of an even number of factors in the numerator and denominator, either or both, of a fraction be reversed, the sign of the fraction is unchanged; but, if the signs of an odd number of factors be reversed, the sign of the fraction is reversed.*

E.g., $$\frac{x-a}{(a-b)(b-c)(c-a)}=-\frac{x-a}{(a-b)(b-c)(a-c)}$$
$$=\frac{x-a}{(b-a)(b-c)(a-c)}$$
$$=\frac{a-x}{(a-b)(b-c)(a-c)}.$$

8. Observe that the sign of a fraction affects each term of the numerator (or each term of the denominator); or, the dividing line between the numerator and the denominator has the same effect as parentheses.

E.g.,
$$-\frac{a-b+c}{d}=-(a-b+c)\div d$$
$$=(-a+b-c)\div d$$
$$=\frac{-a+b-c}{d}.$$

EXERCISES I.

Change each of the following fractions into an equivalent fraction with sign reversed, leaving the denominator unchanged:

1. $-\frac{a-b+c}{x+y-z}$. 2. $\frac{x^2-x-1}{x^2+x-1}$. 3. $\frac{(x-a)(b-c)}{(x-b)(a-c)}$

Change each of the following pairs of fractions into two equivalent fractions whose denominators are equal:

4. $-\frac{a}{a^3-1}, \frac{1}{1-a^3}$. 5. $\frac{1}{a+b-c}, \frac{1}{c-a-b}$.

Change each of the following pairs of fractions into two equivalent fractions whose denominators have a common factor:

6. $\frac{x-a}{(x-b)(x-c)}, \frac{b-x}{(a-x)(c-x)}$.

7. $\frac{a}{(a-b)(a-c)(x-c)}, \frac{b}{(b-c)(c-a)(x-b)}$.

Classification of Fractions.

9. A **Proper Fraction** is one whose numerator is of lower degree than its denominator in a common letter of arrangement.

E.g., $\frac{1}{x+1}, \frac{x-2}{x^2+2x-1}$.

An **Improper Fraction** is one whose numerator is of the same or of a higher degree than its denominator in a common letter of arrangement.

E.g., $\frac{x}{x+1}, \frac{x^3+3x^2+x-1}{x^2+2x-1}$.

A **Fractional Expression** is an expression which has one or more fractional terms.

E.g., $a+\frac{b}{c}, ax+by-\frac{c}{x+y}, \frac{x-y}{a+b}$.

If both integral and fractional terms occur in an expression, it is sometimes called a **Mixed Expression.**

An improper fraction can be reduced to a mixed expression. Thus, if x^3+x^2-4x+3 be divided by x^2+2x-1, the quotient will be $x-1$, and the remainder $-x+2$. Therefore, by Ch. III., § 4, Art. 9,

$$\frac{x^3+x^2-4x+3}{x^2+2x-1}=x-1+\frac{-x+2}{x^2+2x-1}=x-1-\frac{x-2}{x^2+2x-1}.$$

EXERCISES II.

Reduce each of the following fractions to equivalent fractional expressions, containing only proper fractions:

1. $\frac{x^3+x^2-1}{x^2}$.

2. $\frac{x^2-x-1}{x^2}$.

3. $\frac{10a^2-3a+4}{5a^2}$.

4. $\frac{6a^3-9a^2b+5b}{3a}$.

5. $\frac{x^2+x-xy}{x-y}$.

6. $\frac{a^2-b^2-a}{a-b}$.

7. $\frac{9x^2-9x+3}{x-1}$.

8. $\frac{2x^2+x-5}{x+1}$.

9. $\frac{21x^2+20x-1}{3x+2}$.

10. $\frac{m^3-n^3-1}{m-n}$.

11. $\frac{x^3-3x^2+2x-3}{x-1}$.

12. $\frac{m^3-mn^2-m^2n+n^3+1}{m-n}$.

13. $\frac{5x^2-3x-14}{x^2-2}$.

14. $\frac{4x^3+21x+9}{x^2+7}$.

15. $\frac{x^3+x^2-2}{x^2-1}$.

Reduction of Fractions.

10. The reduction of fractions is based upon the following principles:

(i.) *If both numerator and denominator of a fraction be multiplied by one and the same number or expression, not* 0, *the value of the fraction is not changed;* or, stated symbolically,

$$\frac{a}{b}=\frac{am}{bm}.$$

E.g., $\frac{2}{3}=\frac{2\times 5}{3\times 5}=\frac{10}{15}$; $\frac{a-x}{a+x}=\frac{(a-x)\times(a+x)}{(a+x)\times(a+x)}=\frac{a^2-x^2}{(a+x)^2}$.

(ii.) *If both numerator and denominator of a fraction be divided by one and the same number or expression, not* 0, *the value of the fraction is not changed;* or, stated symbolically,

$$\frac{a}{b}=\frac{a\div m}{b\div m}.$$

E.g., $\frac{6}{8}=\frac{6\div 2}{8\div 2}=\frac{3}{4}$; $\frac{a+ab}{a+ac}=\frac{(a+ab)\div a}{(a+ac)\div a}=\frac{1+b}{1+c}$.

Let the fraction $\frac{a}{b}$ be denoted by q, or $q = \frac{a}{b}$. (1)

Multiplying both members of (1) by b, we have, by Ch. I., § 1, Art. 15 (iii.),

$$qb = \frac{a}{b} \times b;$$

or $$qb = a, \text{ by Art. 5.} \qquad (2)$$

Multiplying both members of (2) by m, we have

$$qbm = am. \qquad (3)$$

Dividing both members of (3) by bm, we have, by Ch. I., § 1, Art. 15 (iv.),

$$q = am \div bm = \frac{am}{bm}. \qquad (4)$$

From (1) and (4), we have, by Axiom (iv.),

$$\frac{a}{b} = \frac{am}{bm}.$$

The principle enunciated in (ii.) can be proved in a similar way.

Reduction of Fractions to Lowest Terms.

11. A fraction is said to be *in its lowest terms* when its numerator and denominator have no common integral factor.

E.g., $$\frac{2}{3}, \quad \frac{x-1}{x^2+1}.$$

Ex. 1. Reduce $\frac{6\,a^3b^2c^4}{8\,a^2b^5c^6}$ to its lowest terms.

The factor $2\,a^2b^2c^4$ is the H. C. F. of the numerator and denominator. We, therefore, have

$$\frac{6\,a^3b^2c^4}{8\,a^2b^5c^6} = \frac{6\,a^3b^2c^4 \div 2\,a^2b^2c^4}{8\,a^2b^5c^6 \div 2\,a^2b^2c^4} = \frac{3\,a}{4\,b^3c^2}, \text{ by Art. 10 (ii.).}$$

Ex. 2. $$\frac{a^2 - x^2}{(a+x)^2} = \frac{(a+x)(a-x)}{(a+x)(a+x)} = \frac{a-x}{a+x}.$$

A fraction is reduced to its lowest terms by dividing its numerator and denominator by the H. C. F. of its terms.

This step is called *canceling common factors, and can usually be done mentally, if the terms of the fraction are first resolved into their prime factors.*

EXERCISES III.

Reduce each of the following fractions to its lowest terms:

1. $\frac{ab}{ac}$.
2. $\frac{a^2x}{ax^2}$.
3. $\frac{a^2x^3}{5\,a^3x^2}$.
4. $\frac{4\,x^4m^2n^3}{8\,x^3m^2n^6}$.
5. $\frac{2\,a^2b^3c^4}{5\,a^3b^2c^5}$.
6. $\frac{150\,a^3x^4z^7}{48\,a^4x^7}$.
7. $\frac{5(x+y)^3}{15(x+y)^2}$.
8. $\frac{44(a+c)^r}{66(a+c)^{r-2}}$.
9. $\frac{a^{n+1}b}{a^{n-1}b^m}$.
10. $\frac{m-n}{2\,m-2\,n}$.
11. $\frac{54\,a^nb^{n-2}y^{n+1}}{72\,ab^{n-1}y^n}$.
12. $\frac{a^2+ab}{a^2-ab}$.
13. $\frac{6\,a-9\,b}{8\,a-12\,b}$.
14. $\frac{a^n+a^{n+2}}{a^{n+1}+a^{n+3}}$.
15. $\frac{2-x}{x^2-4}$.
16. $\frac{5\,a^2+5\,ax}{a^2-x^2}$.
17. $\frac{ax+bx}{na^2-nb^2}$.
18. $\frac{3\,x^2-12\,a^2}{3\,x+6\,a}$.
19. $\frac{a-b}{a^3-b^3}$.
20. $\frac{2\,a-3\,b}{8\,a^3-27\,b^3}$.
21. $\frac{x^2+2\,x-3}{x^2+5\,x+6}$.
22. $\frac{x^2-x-12}{(x+3)^2}$.
23. $\frac{5\,x^2+4\,x-1}{5\,x^2+19\,x-4}$.
24. $\frac{ax-ab}{ax+3\,x-3\,b-ab}$.
25. $\frac{x^3-ax^2+b^2x-ab^2}{x^3-ax^2-b^2x+ab^2}$.
26. $\frac{3\,x^2+16\,x-35}{5\,x^2+33\,x-14}$.
27. $\frac{x^4+x^2-2}{x^4+5\,x^2+6}$.
28. $\frac{x^{2n}+2\,x^n+1}{x^{2n}+3\,x^n+2}$.
29. $\frac{bx+2}{2\,b+(b^2-4)x-2\,bx^2}$.
30. $\frac{1-a^2}{(1+ax)^2-(a+x)^2}$.
31. $\frac{x^5-x^4y-xy^4+y^5}{x^4-x^3y-x^2y^2+xy^3}$.
32. $\frac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$.
33. $\frac{a^6+a^4-a^2-1}{a^8-a^6+a^2-1}$.
34. $\frac{n^4-16}{n^4-4\,n^3+8\,n^2-16\,n+16}$.
35. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}$.
36. $\frac{(a+b+c)^2-(a-b-c)^2}{3\,a(b^2+2\,bc+c^2)}$.
37. $\frac{bc(b-c)+ca(c-a)+ab(a-b)}{(b-c)(c-a)(a-b)}$.
38. $\frac{(y-z)^3+(z-x)^3+(x-y)^3}{(x-y)(y-z)(z-x)}$.
39. $\frac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

12. If the numerator and denominator of a fraction cannot be readily factored, we find their H. C. F. by the method of division.

Ex. The H. C. F. of the numerator and denominator of the fraction

$$\frac{3\,x^2-14\,x+16}{6\,x^3-x^2-61\,x+56}$$

is $3\,x-8$. Dividing both terms of the fraction by $3\,x-8$, we have

$$\frac{x-2}{2\,x^2+5\,x-7}.$$

EXERCISES IV.

Reduce the following fractions to their lowest terms:

1. $\dfrac{3x^3 - 8x^2 + 8x - 5}{2x^3 + 5x^2 - 5x + 7}$.

2. $\dfrac{6x^3 + 11x^2 - 6x - 5}{3x^3 + 10x^2 + 3x - 10}$.

3. $\dfrac{x^3 - x^2 + 2}{x^3 - 3x^2 + 4x - 2}$.

4. $\dfrac{2x^3 - 13x^2 + 19x - 20}{2x^3 + 9x^2 - 14x + 24}$.

5. $\dfrac{x^3 - 5x^2 + 13x - 14}{x^3 - x^2 + x + 14}$.

6. $\dfrac{8x^3 + 2x^2 - 5x + 1}{8x^3 + 10x^2 - 11x + 2}$.

7. $\dfrac{x^3 - 3x^2 + 4}{3x^3 - 18x^2 + 36x - 24}$.

8. $\dfrac{2x^3y^3 - 17x^2y^2 + 27xy - 9}{2x^3y^3 - 11x^2y^2 - 15xy + 9}$.

Reduction of Two or More Fractions to a Lowest Common Denominator.

13. Two or more fractions are said to have a common denominator when their denominators are the same.

E.g., $\dfrac{a}{b}$ and $\dfrac{c}{b}$; $\dfrac{x}{a^2 - x^2}$ and $\dfrac{x - y}{(a + x)(a - x)}$.

The **Lowest Common Denominator** (L. C. D.) of two or more fractions is the L. C. M. of their denominators.

E.g., the L. C. D. of $\dfrac{x}{x^2 - 1}$ and $\dfrac{2x}{(x + 1)^2}$ is $(x + 1)^2(x - 1)$, the L. C. M. of $x^2 - 1$ and $(x + 1)^2$.

Ex. 1. Reduce $\dfrac{a}{b^2c}$ and $\dfrac{d}{bc^2}$ to equivalent fractions having a lowest common denominator.

Their required L. C. D. is b^2c^2. Multiplying both terms of $\dfrac{a}{b^2c}$ by $b^2c^2 \div b^2c, = c$, and both terms of $\dfrac{d}{bc^2}$ by $b^2c^2 \div bc^2, = b$, we have

$$\frac{ac}{b^2c^2} \text{ and } \frac{bd}{b^2c^2}.$$

Ex. 2. Reduce $x, = \dfrac{x}{1}$, and $\dfrac{y}{x - y}$ to equivalent fractions having a lowest common denominator.

The required L. C. D. is $x-y$. Multiplying both terms of $\frac{x}{1}$ by $x-y$, and both terms of $\frac{y}{x-y}$ by 1, we have

$$\frac{x^2-xy}{x-y} \text{ and } \frac{y}{x-y}.$$

Ex. 3. Reduce $\frac{1}{x^2-3x+2}, =\frac{1}{(x-1)(x-2)}$, and $\frac{2}{x^2-1}, =\frac{2}{(x-1)(x+1)}$, to equivalent fractions having a lowest common denominator.

The required L. C. D. is $(x-1)(x-2)(x+1)$. Multiplying both terms of the first fraction by

$$(x-1)(x-2)(x+1) \div (x-1)(x-2), \ = x+1,$$

and both terms of the second fraction by

$$(x-1)(x-2)(x+1) \div (x-1)(x+1), \ = x-2,$$

we have $\frac{x+1}{(x-1)(x-2)(x+1)}$ and $\frac{2x-4}{(x-1)(x-2)(x+1)}$.

Ex. 4. Reduce $\frac{1}{x-a}$ and $\frac{1}{a^2-x^2}$ to equivalent fractions having a lowest common denominator.

Observe that the denominator of the first fraction is, except for sign, a factor of the denominator of the second fraction. In such examples we first change the fractions into equivalent fractions whose denominators are arranged to ascending or descending powers of a common letter.

If, in this example, the denominators be arranged to descending powers of a, the first fraction becomes $\frac{-1}{a-x}$, by Art. 7 (i.).

The required L. C. D. is now $a^2-x^2, =(a-x)(a+x)$. Multiplying both terms of $\frac{-1}{a-x}$ by $(a-x)(a+x) \div (a-x), =(a+x)$, and both terms of $\frac{1}{a^2-x^2}$ by 1, we have

$$\frac{-(a+x)}{a^2-x^2}, \ = -\frac{a+x}{a^2-x^2}, \ = \frac{x+a}{x^2-a^2}, \text{ and } \frac{1}{a^2-x^2}.$$

These examples illustrate the following method:

To reduce two or more fractions to equivalent fractions with a lowest common denominator, multiply both numerator and denominator of each fraction by the quotient obtained by dividing their L. C. D. by the denominator of the fraction.

EXERCISES V.

Reduce the following fractions to equivalent fractions having a lowest common denominator:

1. $1-a, \dfrac{a^2}{a+1}.$

2. $m, \dfrac{1+4m}{m-4}.$

3. $\dfrac{15}{14xy^2}, \dfrac{2x}{3y^2}.$

4. $\dfrac{3}{5a^2b}, \dfrac{7}{15abx}, \dfrac{1}{10b^2x}.$

5. $\dfrac{2-3x}{4x}, \dfrac{5+2x}{12x^2}.$

6. $\dfrac{5a-4b}{6a^2b}, \dfrac{3b-2a}{8ab^2}.$

7. $\dfrac{1}{x+2}, \dfrac{5}{3x+6}.$

8. $\dfrac{1}{x^2-49}, \dfrac{3}{4x+28}.$

9. $\dfrac{2}{x}, \dfrac{3}{2x-1}, \dfrac{2x}{4x^2-1}.$

10. $\dfrac{1}{x-3}, \dfrac{3}{x^2-9}, \dfrac{5}{3x+9}.$

11. $\dfrac{5}{x+2}, \dfrac{3}{x^2+x-2}, \dfrac{1}{x^2-4}.$

12. $\dfrac{b}{ax+ab}, \dfrac{a}{x^2-b^2}, \dfrac{c}{bx-ab}.$

13. $\dfrac{x}{x-1}, \dfrac{1}{x+1}, \dfrac{1}{1-x^2}.$

14. $\dfrac{m}{y(x-y)}, \dfrac{y}{m(y-x)}, \dfrac{1+m}{my}.$

15. $\dfrac{ax-b}{ax+ab}, \dfrac{a-bx}{bx+b^2}, \dfrac{1}{a^2b^2}.$

16. $\dfrac{a}{1-a}, \dfrac{1}{a^2-a}, \dfrac{3a+1}{a^2-1}.$

17. $\dfrac{3}{2x-2}, \dfrac{5}{x^2-2x+1}, \dfrac{x}{1-x^2}.$

18. $\dfrac{1}{n-m}, \dfrac{3nm}{n^3-m^3}, \dfrac{m-n}{m^2+mn+n^2}.$

19. $\dfrac{1}{x^2+2x-8}, \dfrac{1}{x^2-5x+6}, \dfrac{2}{2x^2+x-10}.$

20. $\dfrac{3}{x^2+2ax-3a^2}, \dfrac{1}{x^2-9a^2}, \dfrac{4}{x^2+4ax+3a^2}.$

21. $\dfrac{1}{2x^2-4x+2}, \dfrac{1}{2x^2+4x+2}, \dfrac{1}{1-x^2}.$

22. $\dfrac{1}{(a-c)(a-b)}, \dfrac{1}{(b-a)(b-c)}, \dfrac{1}{(c-a)(c-b)}.$

Addition and Subtraction of Fractions.

14. *The sum, or the difference, of two fractions having a common denominator is a fraction whose numerator is the sum, or the difference, of the numerators of the given fractions, and whose denominator is their common denominator;* or, stated symbolically,

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}, \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

E.g., $$\frac{2x}{x-1}-\frac{1+x}{x-1}=\frac{2x-(1+x)}{x-1}=\frac{x-1}{x-1}=1.$$

We have $\frac{a}{c}+\frac{b}{c}=a\div c+b\div c$, by definition of a fraction,

$=(a+b)\div c$, by the Distributive Law,

$=\frac{a+b}{c}$, by definition of a fraction.

In like manner, the principle can be proved for the difference of two fractions.

15. If the fractions to be added or subtracted do not have a common denominator, they should first be reduced to equivalent fractions having a lowest common denominator.

Ex. 1. Simplify $\frac{a}{b^2c}+\frac{d}{bc^2}$.

We have $$\frac{a}{b^2c}+\frac{d}{bc^2}=\frac{ac}{b^2c^2}+\frac{bd}{b^2c^2}=\frac{ac+bd}{b^2c^2}.$$

Ex. 2. Simplify $\frac{2x-5y}{5}-\frac{3x-6y+2z}{4}-\frac{x+4y-6z}{20}.$

Reducing to L. C. D., we have

$$\frac{8x-20y}{20}-\frac{15x-30y+10z}{20}-\frac{x+4y-6z}{20}$$

$$=\frac{8x-20y-15x+30y-10z-x-4y+6z}{20}$$

$$=\frac{-8x+6y-4z}{20}=\frac{-4x+3y-2z}{10}=-\frac{4x-3y+2z}{10}.$$

Observe that the expressions in this example are not *algebraic* fractions.

Ex. 3. Simplify $\dfrac{1}{1-x}-\dfrac{2}{x+1}+\dfrac{3x}{x^2-1}.$

Changing the signs of the terms of the first fraction, we have

$$\frac{-1}{x-1}-\frac{2}{x+1}+\frac{3x}{x^2-1}=\frac{-(x+1)}{x^2-1}-\frac{2(x-1)}{x^2-1}+\frac{3x}{x^2-1}$$

$$=\frac{-(x+1)-2(x-1)+3x}{x^2-1}$$

$$=\frac{1}{x^2-1}.$$

Ex. 4. Simplify $\dfrac{3}{x+4}+\dfrac{4}{x-5}-\dfrac{3}{x-4}-\dfrac{4}{x+5}.$

The character of the denominators in this example suggests that it is better first to unite the first and third fractions, and the second and fourth fractions separately, and then to unite these results. Reducing, in pairs, to L. C. D., we have

$$\frac{3(x-4)}{x^2-16}-\frac{3(x+4)}{x^2-16}+\frac{4(x+5)}{x^2-25}-\frac{4(x-5)}{x^2-25}$$

$$=\frac{3(x-4)-3(x+4)}{x^2-16}+\frac{4(x+5)-4(x-5)}{x^2-25}$$

$$=\frac{-24}{x^2-16}+\frac{40}{x^2-25}$$

$$=\frac{-24(x^2-25)+40(x^2-16)}{(x^2-16)(x^2-25)}=\frac{8(2x^2-5)}{(x^2-16)(x^2-25)}.$$

Ex. 5. Simplify $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}.$

Changing the fractions into equivalent fractions, whose denominators, taken in pairs, have one common factor, we have

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}$$

$$=\frac{b-c}{(a-b)(a-c)(b-c)}-\frac{a-c}{(a-b)(b-c)(a-c)}$$

$$+\frac{a-b}{(a-c)(b-c)(a-b)}=\frac{(b-c)-(a-c)+(a-b)}{(a-b)(a-c)(b-c)}=0.$$

Ex. 6. Simplify $1 - x + x^2 - \frac{x^3}{1+x}$.

We have
$$1 - x + x^2 - \frac{x^3}{1+x} = \frac{(1 - x + x^2)(1+x) - x^3}{1+x}$$
$$= \frac{1 + x^3 - x^3}{1+x} = \frac{1}{1+x}.$$

EXERCISES VI.

Simplify the following expressions:

1. $1 + \frac{1}{x-1}$.

2. $2m - \frac{3m - 5n}{4}$.

3. $a - \frac{a^2}{a+b}$.

4. $3a + \frac{1 - 8a}{3}$.

5. $x - \frac{3x - 4}{3 - x}$.

6. $a^2 + ax + x^2 + \frac{x^3}{a - x}$.

7. $1 + \frac{(a-b)^2}{4ab}$.

8. $1 - \left(a - \frac{a^2}{1+a}\right)$.

9. $a + b - \frac{2ab}{a+b}$.

10. $\frac{1}{x^3} - \frac{x^2 - 1}{x^5}$.

11. $\frac{1-x}{x^n} + \frac{1}{x^{n-1}}$.

12. $\frac{1}{xy} + \frac{1}{xz} - \frac{1}{yz}$.

13. $\frac{b}{2a} + \frac{3b}{4a} - \frac{5b}{6a}$.

14. $\frac{3}{a^n} - \frac{4}{a^{n-1}} + \frac{5}{a^{n-2}}$.

15. $\frac{1}{(a-1)^3} - \frac{2}{(1-a)^3}$.

16. $1 + a + a^2 + \frac{a^3}{1-a}$.

17. $\frac{5ab - 3y}{20y} + \frac{3ab - 5x}{4x}$.

18. $\frac{9a^n}{14b^6c^4} - \frac{5b^{n-4}}{21ac^2} - \frac{2c^{n-5}}{15ab^5}$.

19. $\frac{x}{x-1} + \frac{1}{2x-1}$.

20. $\frac{x-1}{x-2} - \frac{x-3}{x-1}$.

21. $\frac{x-1}{x+1} - \frac{x-2}{x+2}$.

22. $\frac{3}{x-3} - \frac{4}{x+4}$.

23. $\frac{2a + 3x}{2a - 3x} - \frac{2a - 3x}{2a + 3x}$.

24. $\frac{m+n}{m-n} - \frac{m-n}{m+n}$.

25. $\frac{x^m + y^m}{x^m - y^m} - \frac{x^m - y^m}{x^m + y^m}$.

26. $\frac{1+x}{1+x+x^2} + \frac{1-x}{1-x+x^2}$.

27. $\frac{1}{ac + c^2} - \frac{1}{a^2 + ac}$.

28. $\frac{a^2}{ax + x^2} - \left(\frac{a^2 - x^2}{ax} + \frac{x^2}{a^2 + ax}\right)$.

29. $\frac{n}{a^n + 1} - \frac{n}{a^n - 1}$.

30. $\frac{3x - 2}{5} - \frac{x + 7}{2} + 4$.

31. $\frac{a-3}{2} - \frac{a-5}{6} - \frac{4-a}{8}$.

32. $\frac{x-1}{2} - \frac{x-2}{3} + \frac{x+7}{6}$.

33. $\frac{3 - 2a}{3} + \frac{3a - 2}{5} - \frac{6a + 2}{10}$.

34. $\frac{5 - 3x}{4} - \frac{5x - 4}{10} - \frac{25 - 19x}{15}$.

35. $\frac{x-2}{3x} - \frac{2x-5}{4x^2} + \frac{4-3x}{9x}$.

36. $\dfrac{2x-4y}{5}-\dfrac{5x+2y-3z}{10}+\dfrac{x+16y-5z}{15}.$

37. $\dfrac{x-y-z}{4}-\dfrac{5y-3z-x}{7}-\dfrac{5z-10y+6x}{14}.$

38. $\dfrac{x^2-3x+1}{18}-\dfrac{3x^2-2x-4}{12}-\dfrac{6x-3x^2}{16}.$

39. $\dfrac{3a-4b}{7}-\dfrac{2a-b-c}{3}+\dfrac{15a-4c}{12}-\dfrac{a-4b}{21}.$

40. $\dfrac{a}{3-a}-\dfrac{9}{a^2-3a}.$

41. $\dfrac{2}{x}+\dfrac{x-6}{3x+6}-\dfrac{1}{x^2+2x}.$

42. $\dfrac{3a-1}{a^2-9}-\dfrac{1-3a}{a+3}-\dfrac{3a-16}{a-3}.$

43. $\dfrac{x-1}{6x+24}-\dfrac{1-x}{x^2-16}-\dfrac{x-5}{3x-12}.$

44. $\dfrac{a^2+3n^2+4an}{a^2+n^2+2an}-2.$

45. $\dfrac{a-b}{a+b}-\left(\dfrac{a^2-b^2}{a^2+b^2}-\dfrac{a+b}{a-b}\right).$

46. $\dfrac{1}{(x-1)^2}+\dfrac{2}{x-1}-\dfrac{2x}{x^2+1}.$

47. $\dfrac{1}{1+x}+\dfrac{1}{1-x}-\dfrac{2x}{1-x^2}.$

48. $\dfrac{2a}{a^2-1}-\dfrac{1}{a+1}.$

49. $\dfrac{ac}{a^2-4y^2}+\dfrac{bd}{ac+2cy}.$

50. $\dfrac{5xy}{4a^2-9b^2}-\dfrac{xy}{9bd-6ad}.$

51. $\dfrac{m}{m-n}+\dfrac{2mn}{n^2-m^2}-\dfrac{2m}{m+n}.$

52. $\dfrac{3}{2x-1}+\dfrac{7}{2x+1}-\dfrac{4-20x}{1-4x^2}.$

53. $\dfrac{3a}{a+x}+\dfrac{a}{a-x}-\dfrac{2ax}{a^2-x^2}.$

54. $\dfrac{2m-3}{1-4m^2}+\dfrac{3}{1-2m}+\dfrac{2}{m}.$

55. $\dfrac{a-1}{a+1}-\left(\dfrac{a+1}{1-a}+\dfrac{a^2+1}{a^2-1}\right).$

56. $\dfrac{a+x}{a-x}-\left[\dfrac{x-a}{x+a}-\left(\dfrac{a^2+x^2}{a^2-x^2}+\dfrac{4ax}{a^2+x^2}\right)\right].$

57. $\dfrac{a+x}{2a+2x+4}-\dfrac{2}{a^2+2ax+2a+2x+x^2}.$

58. $\dfrac{5a}{9a^2-25b^2}-\dfrac{2a+3b}{6ad+10bd}-\dfrac{4a-b}{6ad-10bd}.$

59. $\dfrac{2}{x^2-3x+2}-\dfrac{3}{x^2-5x+6}+\dfrac{4}{x^2-4x+3}.$

60. $\dfrac{5}{x^2-2x-3}-\dfrac{4}{x^2-9}-\dfrac{7}{x^2+4x+3}.$

61. $\dfrac{4x}{x^2-3ax+2a^2}-\dfrac{3x}{2x^2-3ax+a^2}-\dfrac{5x}{2x^2-5ax+2a^2}.$

62. $\dfrac{1}{a-1}-\dfrac{a^2+2a}{a^3-1}.$

63. $\dfrac{1}{x+1}+\dfrac{x^2+x}{x^3+1}.$

64. $\dfrac{1}{(a-1)^2+3a}-\dfrac{1}{1-a^3}-\dfrac{1}{a-1}.$

65. $\dfrac{a-2}{a^2-a+1}-\dfrac{1}{a+1}+\dfrac{a^2+a+3}{a^3+1}.$

66. $\frac{a-2n}{a^3+n^3}-\frac{a-n}{a^2n-an^2+n^3}-\frac{1}{an+n^2}.$

67. $\frac{1}{n-m}-\frac{3nm}{n^3-m^3}-\frac{m-n}{m^2+mn+n^2}.$

68. $\frac{1}{x^4+x^2+1}-\frac{1}{x-1-x^2}+\frac{1}{x+1+x^2}.$

69. $\frac{1}{2x^2-4x+2}+\frac{1}{2x^2+4x+2}-\frac{1}{1-x^2}.$

70. $\frac{1}{a^2+5a+6}+\frac{2a}{a^2+4a+3}+\frac{1}{(a+1)^2+(a+1)}-\frac{2}{a+3}.$

71. $\frac{x^2+x-1}{x^3-x^2+x-1}+\frac{x^2-x-1}{x^3+x^2+x+1}-\frac{x}{1-x^2}-\frac{2x^3}{x^4-1}.$

72. $\frac{b}{b+x}-\frac{bx}{b^2+x^2}-\frac{x^2}{b^2-x^2}-\frac{2bx^3}{b^4-x^4}.$

73. $\frac{5}{x-2}+\frac{7}{x-1}-\frac{5}{x+2}-\frac{7}{x+1}.$

74. $\frac{4}{x+7}-\frac{1}{x-8}-\frac{4}{x-7}+\frac{1}{x+8}.$

75. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}.$

76. $\frac{ab}{(b-c)(c-a)}+\frac{bc}{(c-a)(a-b)}+\frac{ca}{(a-b)(b-c)}.$

77. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$

78. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}.$

79. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}.$

80. $\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}+\frac{ab}{c(c^2-a^2)(c^2-b^2)}.$

81. $\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2+ac}{(b+c)(b-a)}+\frac{c^2+ab}{(c-a)(c+b)}.$

82. $\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}.$

83. $\frac{2}{a-b}+\frac{2}{b-c}+\frac{2}{c-a}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}.$

84. $\frac{x^2y^2}{b^2c^2}+\frac{(x^2-b^2)(b^2-y^2)}{b^2(c^2-b^2)}+\frac{(c^2-x^2)(c^2-y^2)}{c^2(c^2-b^2)}.$

85. $\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$

Multiplication of Fractions.

16. *The product of two fractions is a fraction whose numerator is the product of the numerators of the given fractions, and whose denominator is the product of their denominators;* or, stated symbolically,

$$\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}.$$

E.g.,

$$\frac{a+x}{b+x} \times \frac{a-x}{b-x} = \frac{a^2-x^2}{b^2-x^2}.$$

We have $\frac{a}{c} \times \frac{b}{d} = (a \div c) \times (b \div d)$, by definition of a fraction

$= a \times b \div c \div d$, by Commutative Law,

$= (ab) \div (cd)$, by Ch. II., § 4, Art. 8 (ii.),

$= \frac{ab}{cd}$, by definition of a fraction.

If the numerator of one fraction and the denominator of another have common factors, such factors should be canceled before the multiplications are performed.

Ex. 1. Simplify $\frac{6(a^2-b^2)}{x^2-y^2} \times \frac{(x+y)^2}{3(a-b)^2}$,

$$= \frac{2 \cdot 3(a+b)(a-b)}{(x+y)(x-y)} \times \frac{(x+y)(x+y)}{3(a-b)(a-b)}.$$

Canceling the common factors, $3(a-b)$, $(x+y)$, we have

$$\frac{2(a+b)}{x-y} \times \frac{x+y}{a-b} = \frac{2(a+b)(x+y)}{(x-y)(a-b)}.$$

Ex. 2. $\frac{2x}{x^2-3x+2} \times \frac{x-1}{x+1} \times \frac{x+1}{x(x^2+1)^2}$,

$$= \frac{2x}{(x-1)(x-2)} \times \frac{x-1}{x+1} \times \frac{x+1}{x(x^2+1)^2}.$$

Canceling common factors, we have

$$\frac{2}{x-2} \times \frac{1}{1} \times \frac{1}{(x^2+1)^2} = \frac{2}{(x-2)(x^2+1)^2}.$$

A mixed expression may be reduced to a fraction before multiplying.

Ex. 3. $\left(1-\frac{b^3}{a^3}\right)\times\left(b^2+a^2-\frac{b^4}{b^2-a^2}\right)=\frac{a^3-b^3}{a^3}\times\frac{b^4-a^4-b^4}{b^2-a^2}$

$$=\frac{a^3-b^3}{a^3}\times\frac{a^4}{a^2-b^2}=\frac{a(a^2+ab+b^2)}{a+b}.$$

17. The principle proved in Ch. III., § 4, Art. 2, namely,

$$a^m \div a^n = a^{m-n}, \text{ when } m > n,$$

can now be extended to the case in which the dividend is a lower power than the divisor.

E.g., $\frac{a^3}{a^5}=\frac{a^3}{a^3\times a^{5-3}}=\frac{1}{a^{5-3}}=\frac{1}{a^2}.$

In general, $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$, when $m < n$.

When $m < n$, we have

$$\frac{a^m}{a^n}=\frac{aaa\cdots\text{ to } m \text{ factors}}{aaa\cdots\text{ to } n \text{ factors}}$$

$$=\frac{aaa\cdots\text{ to } m \text{ factors}}{aaa\cdots\text{ to } m \text{ factors}\times aaa\cdots\text{ to } (n-m) \text{ factors}}$$

$$=\frac{1}{aaa\cdots\text{ to } (n-m) \text{ factors}}=\frac{1}{a^{n-m}}.$$

EXERCISES VII.

Simplify the following expressions:

1. $\frac{7x}{a^3}\times\frac{5ab^3}{14x^2}.$
2. $\frac{15a^3b^2}{22x^2y^5}\times\frac{14xy^2}{25a^2b}.$
3. $\frac{8a^5b^6x^7}{15c^6y^5}\times\frac{5a^4b^8c^2}{6x^8y^8}\times\frac{3c^5y^4}{4a^{18}b^4}.$
4. $\frac{2x-5y}{x+y}\times\frac{2x+5y}{x-y}.$
5. $\frac{5x}{15a-10b}\times(3a-2b).$
6. $\frac{8a^2}{a^2-b^2}\times\frac{a+b}{2a}.$
7. $\frac{ab^2-b^3}{a^2+ab}\times\frac{a^3-ab^2}{2b^3}.$
8. $\frac{x-3}{x+1}\times\frac{x^2+2x+1}{x^3-27}.$
9. $\frac{a(a+b)}{a^2-2ab+b^2}\times\frac{b(a-b)}{a^2+2ab+b^2}.$
10. $\frac{6ax-15bx}{40ay+15dy}\times\frac{8ax+3dx}{4a^2-25b^2}.$
11. $\frac{x^4-y^4}{(x+y)^2}\times\frac{x^2-y^2}{x^2+y^2}\times\frac{x+y}{(x-y)^2}.$
12. $\frac{x^4-y^4}{a^3+b^3}\times\frac{a^2-ab+b^2}{x-y}\times\frac{a+b}{x+y}.$
13. $\frac{x^2-(a+b)x+ab}{x^2-(a+c)x+ac}\times\frac{x^2-c^2}{x^2-b^2}.$
14. $\frac{a^2-(b-c)^2}{x^2-y^2}\times\frac{(x+y)^2}{(a-b)^2-c^2}.$
15. $\frac{x^3-8y^3}{x^2-y^2}\times\frac{x+y}{x^2-2xy+4y^2}.$
16. $\frac{x^2-4}{x^2-8x+15}\times\frac{x^2-9}{x^2-8x+12}.$

17. $\dfrac{4x^2-9y^2}{22a^2-10ab}\times\dfrac{33ab-15b^2}{6ax-9ay}\times\dfrac{12a^2}{10bx+15by}.$

18. $\dfrac{x^2+x-6}{x^2-x-20}\times\dfrac{x^2+x-12}{x^2+x-6}\times\dfrac{x^2-3x-10}{x^2-4}.$

19. $\dfrac{y+x}{(m+n)^3}\times\dfrac{x^2-y^2}{12}\times\dfrac{(m+n)^2}{m-n}\times\dfrac{6(m^2-n^2)}{x+y}.$

20. $(x^2-x+1)\left(\dfrac{1}{x^2}+\dfrac{1}{x}+1\right).$ 21. $\left(\dfrac{a}{b}+1+\dfrac{b}{a}\right)\left(\dfrac{a}{b}-1+\dfrac{b}{a}\right).$

22. $\left(\dfrac{a+x}{a}-\dfrac{x-y}{x}\right)\times\dfrac{a^2}{x^2+ay}.$ 23. $\left(\dfrac{x^2+1}{2x-1}-\tfrac{1}{2}x\right)\times\dfrac{1-2x}{x+2}.$

24. $\dfrac{1-x^2}{1+y}\times\dfrac{1-y^2}{x+x^2}\times\left(1+\dfrac{x}{1-x}\right).$

25. $\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\times\dfrac{x+y}{2y}.$

26. $\left[a^2-ab+b^2-\dfrac{a^3-b^3}{a+b}\right]\left[1+a-\dfrac{a(b-1)}{b}\right].$

27. $\left(x-\dfrac{1}{x}\right)^2.$ 28. $\left(2x+\dfrac{1}{2x}\right)^3.$ 29. $\left(2a+\dfrac{3}{a}\right)^4.$

Powers of Fractions.

18. From the principle for multiplying fractions we have:

A power of a fraction is a fraction whose numerator is the like power of the numerator of the given fraction, and whose denominator is the like power of the denominator; or, stated symbolically,

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n},$$

wherein n is, as yet, a positive integer.

Ex. 1. $$\left(\frac{2a^2b^3}{c^4}\right)^3=\frac{8(a^2)^3(b^3)^3}{(c^4)^3}=\frac{8a^6b^9}{c^{12}}.$$

The converse of the principle evidently holds; that is,

$$\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n.$$

Ex. 2. $$\frac{(x^2-5x+6)^2}{(x-3)^2}=\left(\frac{x^2-5x+6}{x-3}\right)^2=(x-2)^2.$$

EXERCISES VIII.

Simplify the following expressions:

1. $\left(\frac{2ab^2}{3x}\right)^3$. 2. $\frac{(2ab^2)^3}{3x}$. 3. $\left(-\frac{6x^2y}{7b^3c^2}\right)^3$. 4. $\left(\frac{5a^2b^3}{3x^2y^2}\right)^3$.

5. $\left(-\frac{2a^3x^4z^2}{5b^2c^5y}\right)^6$. 6. $\left(\frac{a^2}{b}\right)^n$. 7. $\left(-\frac{2a^3x}{5b^2y^3}\right)^{2n}$. 8. $\left(\frac{2a^2x^{n-1}}{3b^3y^n}\right)^{n+1}$.

9. $\frac{(x^2-1)^5}{(x-1)^5}$. 10. $\frac{(x^3-1)^3}{(x^2+x+1)^3}$. 11. $\frac{(x^4-y^4)^3}{(x^2+y^2)^3}$. 12. $\frac{(x^2-5x+6)^3}{(x-2)^3}$.

13. $\left(\frac{a+1}{b+1}\right)^2 \times \frac{b^2-1}{a^3+1}$. 14. $\frac{(x+7)^4}{(x^2+6x-7)^4}$. 15. $\frac{(x^4-xy^3)^{15}}{(x^3-y^3)^{15}}$.

16. $\left(\frac{a^2+ab+b^2}{x^3+y^3}\right)^3 \times \left(\frac{x^2-xy+y^2}{a^3-b^3}\right)^3$.

17. $\left(\frac{8x^2-10x-3}{x^2-2xy+y^2}\right)^2 \times \left(\frac{x^2-y^2}{2x^2-5x+3}\right)^2$.

18. $\frac{d^3-z^3}{d^3+z^3} \times \frac{d+z}{d-z} \times \left(\frac{d^2-dz+z^2}{d^2+dz+z^2}\right)^2$.

Express each of the following fractions as the square of a fraction:

19. $\frac{x^2}{y^4}$. 20. $\frac{16a^4b^6}{81x^6y^8}$. 21. $\frac{625m^{12}n^{6n}}{a^{2n-2}b^4}$.

Reciprocal Fractions.

19. The **Reciprocal** of a fraction is a fraction whose numerator is the denominator, and whose denominator is the numerator of the given fraction.

E.g., the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

20. The product of a fraction and its reciprocal is 1.

For $$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba} = 1.$$

Division of Fractions.

21. *The quotient of one fraction divided by another is equal to the product of the dividend and the reciprocal of the divisor;* or, stated symbolically,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

E.g., $$\frac{a-x}{b-x} \div \frac{b+x}{a+x} = \frac{a-x}{b-x} \times \frac{a+x}{b+x} = \frac{a^2-x^2}{b^2-x^2}.$$

We have

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= (a \div b) \div (c \div d), \text{ by definition of a fraction,}\\ &= a \div b \div c \times d, \text{ since } \div (c \div d) = \div\, c \times d,\\ &= a \div b \times d \div c, \text{ since } \div\, c \times d = \times\, d \div c,\\ &= (a \div b) \times (d \div c), \text{ since } \times\, d \div c = \times (d \div c),\\ &= \frac{a}{b} \times \frac{d}{c}, \text{ by definition of a fraction.}\end{aligned}$$

Ex. 1. $$\begin{aligned}\frac{4(a^2 - ab)}{(a+b)^2} \div \frac{6a}{a^2 - b^2} &= \frac{4a(a-b)}{(a+b)^2} \times \frac{(a-b)(a+b)}{6a}\\ &= \frac{2(a-b)^2}{3(a+b)}.\end{aligned}$$

Ex. 2. $$\begin{aligned}(a^2 - b^2 - c^2 + 2bc) \div \frac{a+b-c}{a+b+c}\\ &= (a+b-c)(a-b+c) \times \frac{a+b+c}{a+b-c}\\ &= (a-b+c)(a+b+c).\end{aligned}$$

22. If the numerator and denominator of the dividend be multiples of the numerator and denominator of the divisor, respectively, the following principle should invariably be used:

The quotient of one fraction divided by another is a fraction whose numerator is the quotient of the numerator of the first fraction divided by the numerator of the second, and whose denominator is the quotient of the denominator of the first fraction divided by the denominator of the second; or, stated symbolically,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}.$$

E.g., $$\frac{a^2 - x^2}{b^2 - x^2} \div \frac{a - x}{b - x} = \frac{(a^2 - x^2) \div (a - x)}{(b^2 - x^2) \div (b - x)} = \frac{a + x}{b + x}.$$

We have

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= (a \div b) \div (c \div d), \text{ by definition of a fraction,}\\ &= a \div b \div c \times d, \text{ since } \div (c \div d) = \div\, c \times d,\\ &= a \div c \div b \times d, \text{ since } \div\, b \div c = \div\, c \div b,\\ &= (a \div c) \div (b \div d), \text{ since } \div\, b \times d = \div (b \div d),\\ &= \frac{a \div c}{b \div d}, \text{ by definition of a fraction.}\end{aligned}$$

23. Observe that a fraction is divided by an integral expression, which is a factor of its numerator, by dividing its numerator by the expression.

Ex. 1. $\dfrac{a^2-b^2}{xy}\div(a-b)=\dfrac{(a^2-b^2)\div(a-b)}{xy}=\dfrac{a+b}{xy}.$

Also that a fraction is divided by an integral expression, which is not a factor of its numerator, by multiplying its denominator by the expression.

Ex. 2. $\dfrac{a^2+b^2}{xy}\div(a+b)=\dfrac{a^2+b^2}{xy(a+b)}.$

EXERCISES IX.

Simplify the following expressions:

1. $\dfrac{27\,a^3b^4}{16\,x^5y^2}\div\dfrac{9\,a^5b^2}{4\,x^3y^6}.$

2. $\dfrac{a^5b^6}{x^7y^3}\div\dfrac{a^3b^4}{x^5y^6}.$

3. $\dfrac{12\,x^5y^6}{35\,a^7b^3}\div\dfrac{18\,x^6y^5}{7\,a^4b^6}.$

4. $\dfrac{a^{n-1}x^n}{b^3y^{n-1}}\div\dfrac{a^{n+1}x^3}{b^2y^n}.$

5. $\dfrac{x^2+7\,x+12}{x^2+2\,x-15}\div\dfrac{x+4}{x+5}.$

6. $\dfrac{6(a^2-b^2)^2}{7(x^3-1)}\div\dfrac{3(a+b)}{(1-x)}.$

7. $\dfrac{2\,a^3-2\,ab^2}{a+2\,b}\div\dfrac{a^2-b^2}{2\,a+4\,b}.$

8. $\dfrac{x^2-6\,x+8}{x^2+2\,x+1}\div\dfrac{x-4}{x+1}.$

9. $\dfrac{x^2+y^2-2\,xy-z^2}{a^2-9+4\,b^2+4\,ab}\div\dfrac{x-y+z}{a+2\,b-3}.$

10. $\dfrac{45\,dx-9\,dy}{20\,abx-10\,b^2x}\div\dfrac{30\,x-6\,y}{20\,a^2x-5\,b^2x}.$

11. $\dfrac{a^2-(b-c)^2}{(a^2-b^2)^2}\div\dfrac{a-b+c}{a^4-b^4}.$

12. $\dfrac{x^3-1}{x^2-a^2}\div\dfrac{x^2+x+1}{x-a}.$

13. $\dfrac{1+n-n^3-n^4}{1-a^2}\div\dfrac{n^2-1}{a^2-1}.$

14. $\dfrac{1-2\,x}{1-x^3}+\dfrac{1-2\,x+x^2-2\,x^3}{1+2\,x+2\,x^2+x^3}.$

15. $\dfrac{a^2+ab}{a^2+b^2}\div\dfrac{a^3b+ab^3+2\,a^2b^2}{a^4-b^4}.$

16. $\dfrac{1-x}{x^3+x^4-x^5}+\dfrac{1-x^3}{x^5-x^3-2\,x^2--x}.$

17. $\dfrac{(a+2\,b)a^3-(2\,a+b)b^3}{a^4b^4}\div\dfrac{(a+b)^2}{a^4b^2+a^2b^4}.$

18. $\dfrac{x^2+2\,x-3}{x^2-2\,x-3}\div\dfrac{x^2+4\,x+3}{x^2-4\,x+3}\times\dfrac{x^3+1}{x^3-1}.$

19. $\dfrac{x^4+x^2y^2+y^4}{x^2+y^2}\times\dfrac{x^2+y(2\,x+y)}{x^3-y^3}\div\dfrac{x^3+y^3}{x^2-y(2\,x-y)}.$

Complex Fractions.

24. A **Complex Fraction** is a fraction whose numerator and denominator, either or both, are fractional expressions.

E.g.,
$$\frac{\frac{2}{3}}{\frac{4}{5}},\quad \frac{\frac{a+x}{a-x}}{\frac{a+y}{a-y}},\quad \frac{1+\frac{1}{x}}{1-\frac{1}{x}}.$$

Observe that the line which separates the terms of the complex fraction is drawn heavier than the lines which separate the terms of the fractions in its numerator and denominator.

If no distinction be made between the lines of division, the indicated divisions are to be performed successively from above downward.

E.g., $\begin{array}{c}2\\\hline 3\\\hline 4\\\hline 5\end{array} = 2 \div 3 \div 4 \div 5 = 2 \div (3 \times 4 \times 5)$, by Ch. II., § 4, Art. 8,

$= 2 \div 60 = \frac{1}{30}$;

while
$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \times \frac{5}{4} = \frac{5}{6}.$$

25. Complex fractions are simplified by applying successively the principles already established for simple fractions.

Ex. 1.
$$\frac{\frac{1-x^2}{x}}{1-x} = \frac{1-x^2}{x} \div (1-x) = \frac{1+x}{x} \text{ by Art. 23.}$$

Ex. 2.
$$\frac{\frac{m^2+n^2}{n} - m}{\frac{1}{n} - \frac{1}{m}} \div \frac{m^3+n^3}{m^2-n^2} = \frac{\frac{m^2+n^2-mn}{n}}{\frac{m-n}{mn}} \times \frac{m^2-n^2}{m^3+n^3}$$
$$= \frac{m(m^2+n^2-mn)}{m-n} \times \frac{(m+n)(m-n)}{(m+n)(m^2-mn+n^2)} = m.$$

We might have simplified the complex fraction in Ex. 2 by first multiplying both its terms by *mn*, the L. C. D. of the fractions in them, instead of uniting these fractions before multiplying by mn. This fraction would then first have reduced to

$$\frac{m^3 + mn^2 - m^2n}{m - n}, = \frac{m(m^2 + n^2 - mn)}{m - n}, \text{ as above.}$$

Continued Fractions.

26. A **Continued Fraction** is a fraction whose numerator is an integer, and whose denominator is an integer plus (or minus) another fraction whose numerator is an integer, and whose denominator is an integer plus (or minus) a third fraction, etc.

Ex. 1. $$\frac{1}{2 + \frac{3}{4 - \frac{6}{7}}} = \frac{1}{2 + \frac{3}{\frac{22}{7}}} = \frac{1}{2 + \frac{21}{22}} = \frac{1}{\frac{65}{22}} = \frac{22}{65}.$$

Observe that in this reduction the work proceeds from below upward.

Ex. 2. $$\frac{3}{x + \frac{1}{1 + \frac{x+1}{3-x}}} = \frac{3}{x + \frac{1}{\frac{4}{3-x}}} = \frac{3}{x + \frac{3-x}{4}}$$

$$= \frac{3}{\frac{3x+3}{4}} = \frac{4}{x+1}.$$

EXERCISES X.

Simplify the following expressions:

1. $\dfrac{a + \dfrac{a^2}{c}}{b + \dfrac{bc}{a}}.$

2. $\dfrac{a - \dfrac{ax}{a+x}}{a + \dfrac{ax}{a-x}}.$

3. $\dfrac{\dfrac{x}{x-1} - \dfrac{x+1}{x}}{\dfrac{x}{x+1} - \dfrac{x-1}{x}}.$

4. $\dfrac{\dfrac{a}{a-1} + 1}{1 - \dfrac{a}{1-a}}.$

5. $\dfrac{x}{1 - \dfrac{1}{1+x}}.$

6. $a + \dfrac{a}{a + \dfrac{1}{a}}.$

7. $\dfrac{x - \dfrac{1}{1+x}}{\dfrac{1 - x - x^2}{x+1}}.$

8. $\dfrac{\dfrac{a^2}{b^3} - \dfrac{b^2}{a^3}}{1 - \dfrac{b}{a}}.$

9. $x - \dfrac{x}{1 + x + \dfrac{2x^2}{1-x}}.$

10. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$. 11. $\dfrac{\dfrac{x^2+1}{2x-1}-\dfrac{1}{2}x}{\dfrac{x+2}{1-2x}}$. 12. $\dfrac{\dfrac{a+x}{x}-\dfrac{2x}{x-a}}{\dfrac{a^2+x^2}{x-a}}$.

13. $\dfrac{\dfrac{a+x}{a}-\dfrac{x-y}{x}}{\dfrac{x^2+ay}{a^2}}$. 14. $\dfrac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$. 15. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}$.

16. $\dfrac{\dfrac{n}{n+x}-\dfrac{n}{n-x}}{\dfrac{n}{n-x}+\dfrac{n}{n+x}}$. 17. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{4ax}{a^2-x^2}}$. 18. $\dfrac{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}$.

19. $\dfrac{x^2}{a+\dfrac{x^2}{a+\dfrac{x^2}{a}}}$. 20. $x+1-\dfrac{x}{x+2-\dfrac{x+1}{x+\dfrac{1}{x+2}}}$.

21. $\dfrac{\dfrac{x}{x-2}-\dfrac{x}{x+2}}{\dfrac{2x}{\frac{1}{2}x^4-x^3+4x-8}}$. 22. $\dfrac{\dfrac{x^4}{x+1}-\dfrac{1}{x^4+x^5}}{x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}}$.

23. $\dfrac{\dfrac{a}{n}-\dfrac{n-x}{a}+\dfrac{ax}{n^2-nx}}{\dfrac{a}{n-x}+\dfrac{n-x}{a}+2}$. 24. $\dfrac{\left(p+\dfrac{1}{q}\right)^p\left(p-\dfrac{1}{q}\right)^q}{\left(q+\dfrac{1}{p}\right)^p\left(q-\dfrac{1}{p}\right)^q}$.

25. $\dfrac{\dfrac{a+2b}{ab^4}-\dfrac{2a+b}{a^4b}}{\dfrac{b^2+c^2}{b^2c^2}\left(\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)-\left(\dfrac{1}{a^2}-\dfrac{1}{c^2}\right)\dfrac{a^2+c^2}{a^2c^2}}$.

26. $\dfrac{\dfrac{a^2+ab}{a^2+b^2}}{\dfrac{a^3b+ab^3+2a^2b^2}{a^4-b^4}}\times\dfrac{\dfrac{a^4-a-3a^3+3a^2}{a^3b-b^4}}{\dfrac{a^4+a^2-2a^3}{a^2b^2+ab^3+b^4}}$.

Factors of Fractional Expressions.

27. Fractional Expressions can be factored by the same methods as were employed in factoring integral expressions.

Ex. 1.
$$k^2+k+n+\frac{n}{k}=k^2\left(1+\frac{1}{k}\right)+n\left(1+\frac{1}{k}\right)$$
$$=\left(1+\frac{1}{k}\right)\left(k^2+n\right).$$

Ex. 2.
$$\frac{4a^2}{9b^2}-1=\left(\frac{2a}{3b}+1\right)\left(\frac{2a}{3b}-1\right).$$

Ex. 3.
$$x^2+\frac{1}{x^2}+1=x^2+2+\frac{1}{x^2}-1=\left(x+\frac{1}{x}\right)^2-1$$
$$=\left(x+\frac{1}{x}+1\right)\left(x+\frac{1}{x}-1\right).$$

EXERCISES XI.

Factor the following expressions:

1. $a^2b^2-\frac{1}{c^2d^2}$
2. $x^3+\frac{1}{x^3}$.
3. $\frac{64x^3}{125y^3}-1$.
4. $x^2+\frac{1}{4x^2}$.
5. $x^2+1+\frac{1}{4x^2}$.
6. $4x^2+2\frac{x}{y}+\frac{1}{4y^2}$.
7. $x^3+\frac{1}{x^3}+3x+\frac{3}{x}$.
8. $x^3+x+\frac{1}{x}+\frac{1}{x^3}$.
9. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+1$.
10. $x^2+x+2+\frac{1}{x}+\frac{1}{x^2}$.
11. $x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$.
12. $x^5-\frac{1}{x^5}+x-\frac{1}{x}$.

Indeterminate Fractions.

28. By Art. 5, a fraction is a number which, multiplied by the denominator, gives the numerator. Therefore the fraction $\frac{0}{0}$ is a number which, multiplied by 0, gives 0. But by Ch. III., § 3, Art. 16, *any* number, multiplied by 0, gives 0.

Therefore the fraction $\frac{0}{0}$ may denote *any* number whatever. For this reason, it is called an **Indeterminate Fraction.**

Some Principles of Fractions.

29. The following principles will be of use in subsequent work:

(i.) *If*
$$\frac{n_1}{d_1}=\frac{n_2}{d_2}=\frac{n_3}{d_3}=\text{etc.},$$
then each of these fractions is equal to the fraction
$$\frac{an_1+bn_2+cn_3+\cdots}{ad_1+bd_2+cd_3+\cdots}.$$

E.g.,
$$\frac{2}{3}=\frac{4}{6}=\frac{5\times 2+6\times 4}{5\times 3+6\times 6}.$$

Let the common value of the given fractions be v. Then from
$$\frac{n_1}{d_1}=v,\quad \frac{n_2}{d_2}=v,\quad \frac{n_3}{d_3}=v,\ \text{etc.},$$
we have
$$n_1=d_1v,\ n_2=d_2v,\ n_3=d_3v,\ \text{etc.}$$

Multiplying these equations by a, b, c, etc., respectively, and adding corresponding members of the resulting equations, we have

$$an_1 + bn_2 + cn_3 + \cdots = ad_1v + bd_2v + cd_3v + \cdots$$
$$= (ad_1 + bd_2 + cd_3 + \cdots)v.$$

Therefore

$$\frac{an_1 + bn_2 + cn_3 + \cdots}{ad_1 + bd_2 + cd_3 + \cdots} = v = \frac{n_1}{d_1} = \frac{n_2}{d_2} = \text{etc.}$$

(ii.) *In particular,*

$$\frac{n_1}{d_1} = \frac{n_2}{d_2} = \frac{n_3}{d_3} = \cdots = \frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots}.$$

E.g.,

$$\frac{2}{3} = \frac{4}{6} = \frac{2+4}{3+6}.$$

(iii.) *If the fraction $\frac{n}{d}$ be in its lowest terms, then $\frac{n^p}{d^p}$, wherein p is a positive integer, is in its lowest terms.*

For by Ch. VIII., § 2, Art. 13 (vii.), n^p and d^p are prime to each other when n and d are prime to each other.

(iv.) *If two fractions, $\frac{n}{d}$ and $\frac{N}{D}$, whose terms are positive integers, be equal, and if $\frac{n}{d}$ be in its lowest terms, then $N = kn$, $D = kd$, wherein k is a positive integer.*

E.g., $\frac{10}{15} = \frac{2}{3}$, and $10 = 5 \times 2$, $15 = 5 \times 3$.

From $\frac{N}{D} = \frac{n}{d}$, we have $N = \frac{nD}{d}$. (1)

Since N is an integer, this equation shows that nD is exactly divisible by d; that is, that it contains d as a factor. Also, since $\frac{n}{d}$ is in its lowest terms, n and d are prime to each other. Therefore, by Ch. VIII., § 2, Art. 13 (iv.), d is a factor of D; that is, $D = kd$, wherein k is a positive integer. Substituting kd for D in (1), we have

$$N = \frac{nkd}{d} = kn.$$

(v.) *If two fractions $\frac{n}{d}$ and $\frac{N}{D}$, whose terms are positive integers, be equal, and each be in its lowest terms, then $N = n$ and $D = d$.*

This principle follows directly from (iv.).

EXERCISES XII.

MISCELLANEOUS EXAMPLES.

Simplify the following expressions:

1. $\dfrac{1-\left(\dfrac{1-a}{1+a}\right)^2}{1+\left(\dfrac{1-a}{1+a}\right)^2}.$

2. $\dfrac{(a-b)^2-\left(\dfrac{a^2+b^2}{a+b}\right)^2}{b-a+\dfrac{a^2}{a+b}}.$

3. $\dfrac{a^2-x^2}{a+b}\times\dfrac{a^2-b^2}{ax+x^2}\left(a+\dfrac{ax}{a-x}\right).$

4. $\left(1+a-\dfrac{a^2+3}{a-1}\right)(1-a^2).$

5. $\dfrac{a+b}{ab}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)-\dfrac{b+c}{bc}\left(\dfrac{1}{c}-\dfrac{1}{b}\right).$

6. $\left(\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}\right)\div\left(\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}\right).$

7. $a+b-\dfrac{1}{a+\dfrac{1}{b}}-\dfrac{1}{b+\dfrac{1}{a}}.$

8. $\dfrac{a}{1+\dfrac{a}{b}}+\dfrac{b}{1+\dfrac{b}{a}}-\dfrac{2}{\dfrac{1}{a}+\dfrac{1}{b}}.$

9. $m-\dfrac{1}{1-m+m^2+\dfrac{m^3}{1+m}}.$

10. $\left(\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right)\div\left(\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right)$

11. $\left(\dfrac{x}{a+x}+a\right)\left(\dfrac{a}{a-x}-x\right)-\left(\dfrac{a}{a+x}+x\right)\left(\dfrac{x}{a-x}-a\right).$

12. $\dfrac{1}{1-\dfrac{x}{x-1}}-\dfrac{1}{\dfrac{x}{x+1}-1}.$

13. $\dfrac{a^2-x^2}{\dfrac{1}{a^2}-\dfrac{2}{ax}+\dfrac{1}{x^2}}\times\dfrac{\dfrac{1}{a^2x^2}}{a+x}.$

14. $\dfrac{x-\dfrac{1}{x^2}}{x+\dfrac{1}{x}-2}\div\dfrac{\left(x+\dfrac{1}{x}\right)^2-1}{\left(1-\dfrac{1}{x}\right)\left(x-1+\dfrac{1}{x}\right)}.$

15. $\left(\dfrac{2x+y}{x+y}+\dfrac{2y-x}{x-y}-\dfrac{x^2}{x^2-y^2}\right)\div\dfrac{x^2+y^2}{x^2-y^2}.$

16. $\dfrac{3x^2+3xy}{4xy+6ay}\times\left(\dfrac{x}{ax+ay}+\dfrac{3}{2x+2y}\right).$

17. $\left(\dfrac{n-1}{n+1}-\dfrac{n+1}{n-1}\right)\times\left(\dfrac{1}{2}-\dfrac{n}{4}-\dfrac{1}{4n}\right).$

18. $\dfrac{\dfrac{a^2}{a+n}-\dfrac{a^3}{a^2+n^2+2an}}{\dfrac{a}{a+n}-\dfrac{a^2}{a^2-n^2}}.$

19. $\dfrac{\dfrac{ab+1}{b}}{a+\dfrac{1}{\dfrac{bc+1}{c}}}-\dfrac{1}{b(abc+a+c)}.$

20. $$\frac{a+\dfrac{ab}{c+d}}{c+\dfrac{cd}{a+b}}\times\frac{a-b+d}{b+c+d}\div\frac{a+b}{c+d}.$$

21. $$\frac{a+\dfrac{1}{b}}{b+\dfrac{1}{a}}\times\frac{b+\dfrac{1}{c}}{c+\dfrac{1}{b}}\times\frac{c+\dfrac{1}{a}}{a+\dfrac{1}{c}}.$$

22. $$\frac{\dfrac{1}{x}-\dfrac{x+a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a+x}{a^2+x^2}}+\frac{\dfrac{1}{x}-\dfrac{x-a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}}.$$

23. $$\frac{\dfrac{a^2+ax}{2x}}{a^2-x^2}\times\left(\frac{(a+x)^2}{4ax}-1\right).$$

24. $$\left[\frac{1}{p^2}+\frac{1}{q^2}+\frac{2}{p+q}\left(\frac{1}{p}+\frac{1}{q}\right)\right]\div(p+q)^2.$$

25. $$\left[\left(\frac{x^2}{y^3}+\frac{1}{x}\right)\div\left(\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}\right)\right]\times\frac{-y}{x+y}.$$

26. $$\left[\left(\frac{2x}{x^2+1}+\frac{2x}{x^2-1}\right)\div\left(\frac{x}{x^2+1}-\frac{x}{x^2-1}\right)\right]^2.$$

27. $$\left[(a^2-b^2)\div\left(\frac{1}{b}-\frac{1}{a}\right)\right]-\left[(a^2-b^2)\div\left(\frac{1}{b}+\frac{1}{a}\right)\right].$$

28. $$\left[\left(\frac{1}{a}+\frac{1}{b+c}\right)\div\left(\frac{1}{a}-\frac{1}{b+c}\right)\right]\times\left(1+\frac{b^2+c^2-a^2}{2bc}\right).$$

29. $$\frac{x^3y-y^4}{xy^2+x^2y}\div\left\{\frac{x^4+x^3y+x^2y^2}{(x^2-y^2)^3}\div\left(1+\frac{y}{x}\right)^2\right\}.$$

30. $$-\frac{a^2-1}{n^2-n}\times\left[1-\frac{1}{1-\dfrac{1}{n}}\right]\times\frac{1+n-n^3-n^4}{1-a^2}.$$

31. $$\frac{\dfrac{x-1}{3x+(x-1)^2}-\dfrac{1-3x+x^2}{x^3-1}-\dfrac{1}{x-1}}{\dfrac{1-2x+x^2-2x^3}{1+2x+2x^2+x^3}}.$$

32. $$\frac{1+x+x^2+\cdots+x^{n-1}+\dfrac{x^n}{1-x}}{1+2x+x^2-\dfrac{x+2}{x^2-1}\cdot x^3}.$$

33. $$\frac{\left(x^n-1-\dfrac{7-x^n}{3+x^n}\right)\times\dfrac{4}{x^{n+2}+3x^2}}{\dfrac{6x^{2n}-24}{x^{2n+3}+6x^{n+3}+9x^3}\times\dfrac{2x}{3x^n+6}}.$$

In each of the following expressions make the indicated substitution, and simplify the result:

34. In $\left(\frac{m-a}{m-b}\right)^3$, let $m=\frac{a-b}{2}$.

35. In $\frac{ax^2}{n-cx}+\frac{ax}{c}$, let $x=\frac{n}{c^2-n}$.

36. In $1+\frac{b^2+c^2-a^2}{2bc}$, let $a+b+c=2s$.

37. In $\frac{m}{n}\left(1-\frac{m}{a}\right)+\frac{n}{m}\left(1-\frac{n}{a}\right)$, let $a=m+n$.

38. In $\frac{ax+1}{x}-\left[a(x+1)-\frac{a(x^2-1)-x}{x}\right]$, let $x=\frac{a-1}{a}$.

Verify each of the following identities:

39. $\frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x$, when $x=a-b$.

40. $\frac{a(x-a)}{b+c}+\frac{b(x-b)}{a+c}+\frac{c(x-c)}{a+b}=x$, when $x=a+b+c$.

41. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}$, when $x=\frac{ab}{a+b}$.

42. $(1+x)(1+y)(1+z)=(1-x)(1-y)(1-z)$, when

$$x=\frac{a-b}{a+b},\quad y=\frac{b-c}{b+c},\quad z=\frac{c-a}{c+a}.$$

43. $b^2-x^2=\frac{4}{c^2}s(s-a)(s-b)(s-c)$, when

$$x=\frac{b^2+c^2-a^2}{2c} \text{ and } a+b+c=2s.$$

CHAPTER X.

FRACTIONAL EQUATIONS IN ONE UNKNOWN NUMBER.

1. A **Fractional Equation** is an equation whose members, either or both, are fractional expressions in the unknown number or numbers.

E.g., $\frac{3}{x+2}=\frac{2}{x+1}$, $x-2+\frac{4-2x}{x+1}=0$.

Observe that we cannot speak of the *degree* of a fractional equation. The term *degree*, as used in Ch. IV., § 2, Art. 5, applies only to *integral* equations.

2. The principles of equivalent equations established in Ch. IV., § 3, hold also for fractional equations.

Ex. 1. If both members of the equation

$$\frac{3}{x+2}=\frac{2}{x+1} \tag{1}$$

be multiplied by $(x+2)(x+1)$, the L. C. D. of the denominators of its fractional terms, we obtain the integral equation

$$3(x+1)=2(x+2); \tag{2}$$

whence $x=1$.

The root 1 of the derived equation (2) is found, by substitution, to be a root of the given equation.

Ex. 2. If both members of the equation

$$\frac{-2x^2}{x^2-1}+\frac{x}{1-x}=-\frac{x}{x+1}-3 \tag{1}$$

be multiplied by x^2-1, we obtain the integral equation

$$-2x^2-x(x+1)=-x(x-1)-3(x^2-1),$$

or $$(x+1)(x-3)=0. \tag{2}$$

Now observe that it was not necessary to multiply by x^2-1, $=(x+1)(x-1)$, to clear the given equation of fractions. For, if the terms in the second member be transferred to the first member, we have

$$\frac{-2x^2}{x^2-1}+\frac{x}{1-x}+\frac{x}{1+x}+3=0,$$

or, uniting terms, $$\frac{x^2-2x-3}{x^2-1}=0,$$

or, canceling $x+1$, $$\frac{x-3}{x-1}=0.$$

Clearing the last equation of fractions, we have

$$x-3=0; \tag{3}$$

whence $$x=3.$$

The root 3 of the derived equation (3) is found, by substitution, to be a root of the given equation. Had we solved equation (2), we should have obtained the additional root -1, which, as will be proved later, is not a root of the given equation.

This root, which does not satisfy the given equation, and which *was introduced by multiplying both members of the given equation by the unnecessary factor* $x+1$, is a root of the equation obtained by equating this factor to 0.

3. The roots of a fractional equation are found by solving the integral equation derived from it by clearing of fractions. The equivalence of the given equation and the derived integral equation is determined by the following principle:

If both members of a fractional equation, in one unknown number, be multiplied by an integral expression which is necessary to clear the equation of fractions, the integral equation thus derived will be equivalent to the given fractional equation.

Thus, in Art. 2, Ex. 1, equation (2) is equivalent to (1); and in Ex. 2, equation (3) is equivalent to (1), while (2) is not equivalent to (1).

Let $$\frac{N}{D}=0 \qquad (1)$$

be the given fractional equation when all its terms are transferred to the first member, added algebraically, and the resulting fraction reduced to its lowest terms. In deriving equation (1) from the given fractional equation, terms were transferred from one member to the other, by Ch. IV., § 3, Art. 7 (i.); and then only indicated operations were performed. Therefore equation (1) is equivalent to the given fractional equation.

Clearing (1) of fractions, we have the integral equation

$$N=0. \qquad (2)$$

Any root of (1) reduces $\frac{N}{D}$ to 0. But any value of x which reduces $\frac{N}{D}$ to 0 must reduce N to 0 (Ch. III., § 4, Art. 7), and hence is a root of the derived equation. That is, no solution is lost by the transformation.

Any root of the derived equation reduces N to 0. But, since $\frac{N}{D}$ is a fraction in its lowest terms, N and D have no common factor, and therefore cannot both reduce to 0 for the same value of x (Ch. VIII., § 4. Art. 2). Consequently, any value of x which reduces N to 0 must reduce $\frac{N}{D}$ to 0 (Ch. III., § 4, Art. 6). That is, no root is gained by the transformation.

Therefore the derived integral equation is equivalent to equation (1). and hence to the given fractional equation.

4. If all the terms of a fractional equation be transferred to its first member, be united into a single fraction, and this fraction be reduced to its lowest terms, the integral equation obtained by then clearing of fractions is equivalent to the given fractional equation. But it is not necessary, nor advisable, to make this transformation before clearing of fractions. If any new root be introduced, it will, as we have seen, be a root of one of the factors of the L. C. D., equated to 0, and can therefore be rejected at sight.

Ex. Solve the equation $\frac{7x+10}{x-2}=\frac{5x}{12}+\frac{35}{6}$.

Multiplying by $12(x-2)$,

$$84x+120=5x^2-10x+70x-140.$$

Transferring, uniting terms, and factoring,

$$(x-10)(5x+26)=0.$$

Whence $x = 10$ and $x = -\frac{26}{5}$.

Since neither 10 nor $-\frac{26}{5}$ is a root of the L. C. D. equated to 0, that is, of $12(x-2)=0$, both 10 and $-\frac{26}{5}$ are roots of the given equation.

5. The following suggestions will simplify the work of solving many fractional equations:

(i.) *If any fraction be not in its lowest terms it should be reduced.*

(ii.) *The form of an equation will frequently suggest grouping and uniting some fractional terms. This reduction should always be made if two or more fractions have a common denominator.*

Ex. 1. Solve the equation $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}$.

Uniting the fractional terms in each member separately, and dividing by -2,

$$\frac{1}{(x-2)(x-4)}=\frac{1}{(x-6)(x-8)}.$$

Clearing of fractions,

$$(x-6)(x-8)=(x-2)(x-4).$$

Therefore $x^2-14x+48=x^2-6x+8$.

Whence $-8x=-40$, or $x=5$.

Since 5 is not a root of any factor of the L. C. D. equated to 0, it is a root of the given equation.

Ex. 2. Solve the equation $\frac{x^2}{x-1}=\frac{1}{x-1}+10$. (1)

Transferring and uniting terms, $\frac{x^2-1}{x-1}=10$.

Reducing to lowest terms, $x+1=10$. (2)

Whence $x=9$.

The integral equation (2) is equivalent to the given equation, and therefore 9 is the required root.

Had the given equation been cleared of fractions by multiplying by $x-1$, the root 1 of $x-1=0$ would have been introduced.

(iii.) *An improper fraction should in some cases first be reduced to a mixed expression.*

Ex. 3. Solve the equation $\dfrac{x-2}{x-4}+\dfrac{x-3}{x-5}=2.$

Reducing the improper fractions, we obtain

$$1+\frac{2}{x-4}+1+\frac{2}{x-5}=2,$$

or

$$\frac{1}{x-4}+\frac{1}{x-5}=0.$$

Clearing of fractions, $x-5+x-4=0.$ Whence $x=\frac{9}{2}.$

EXERCISES I.

Solve the following equations:

1. $\dfrac{12}{x}=4.$
2. $5-\dfrac{3}{x}=2.$
3. $\dfrac{5x-5}{x+1}=3.$
4. $\dfrac{x-1}{x+1}=\dfrac{2}{3}.$
5. $\dfrac{x-1}{x-3}=\dfrac{x-4}{x-2}.$
6. $\dfrac{1}{5-\dfrac{1}{x}}=\dfrac{2}{7}.$
7. $\dfrac{x-2}{2x-5}=\dfrac{x-5}{2x-2}.$
8. $\dfrac{25}{x-\frac{7}{3}}-\dfrac{10}{3x-4}=0.$
9. $\dfrac{x}{x+1}=\dfrac{3x}{x+2}-2.$
10. $\dfrac{x-1}{x+1}+\dfrac{1}{x}=1.$
11. $\dfrac{2x+1}{(x+2)^2}+\dfrac{2x+1}{x+2}=2.$
12. $\dfrac{5x}{3x+1}-\dfrac{1}{9x+3}=\dfrac{7}{6}.$
13. $\dfrac{1}{2}+\dfrac{2}{x+2}=\dfrac{13}{8}-\dfrac{5x}{4x+8}.$
14. $\dfrac{3}{4}-\dfrac{\frac{3}{4}x+\frac{3}{4}}{\frac{3}{4}+x}=\dfrac{\frac{3}{4}}{\frac{3}{4}+x}-\dfrac{3}{4}.$
15. $\dfrac{4}{(x+1)^2}+\dfrac{4}{x(x+1)^2}=\dfrac{5}{2x(x+1)}.$
16. $\dfrac{3}{1-3x}+\dfrac{5}{1-5x}=-\dfrac{4}{2x-1}.$
17. $\dfrac{x-7}{x+7}-\dfrac{2x-15}{2x-6}=-\dfrac{1}{2(x+7)}.$
18. $\dfrac{2x+1}{2x-1}-\dfrac{8}{4x^2-1}=\dfrac{2x-1}{2x+1}.$
19. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x+6}.$
20. $\dfrac{x-3}{x^2-9}-\dfrac{12-2x}{x^2-36}=\dfrac{3x-27}{x^2-81}.$
21. $\dfrac{2x+19}{5x^2-5}-\dfrac{17}{x^2-1}-\dfrac{3}{1-x}=0.$

22. $\frac{7}{x^2-1}+\frac{8}{x^2-2x+1}=\frac{x^2-9x}{x^3-x^2-x+1}.$

23. $\frac{7}{6x+30}+\frac{3}{4x-20}=\frac{13}{2x^2-50}.$

24. $\frac{x^2-6}{x^3+8}+\frac{4}{5x^2-10x+20}-\frac{1}{x+2}=0.$

25. $\frac{1}{x^2+2x+1}+\frac{4}{x+2x^2+x^3}=\frac{6}{2x+2x^2}.$

26. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}=\frac{6}{x^4-1}.$

27. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}.$ 28. $\frac{6}{x-5}-\frac{9}{x-8}=\frac{1}{x-7}-\frac{4}{x-4}.$

29. $\frac{7}{x-9}+\frac{2}{x-4}=\frac{7}{x-7}+\frac{2}{x-11}.$ 30. $\frac{1}{x-13}-\frac{2}{x-15}+\frac{2}{x-18}-\frac{1}{x-19}.$

31. $\frac{x+8}{x-5}-\frac{x+6}{x-6}+\frac{x+4}{x-7}=\frac{x+5}{x-5}-\frac{x+2}{x-6}+\frac{x+3}{x-7}.$

32. $\frac{x+5}{x-7}-\frac{x+3}{x-8}+\frac{x+1}{x-9}=\frac{x+2}{x-7}-\frac{x+1}{x-8}+\frac{x}{x-9}.$

Problems.

6. **Pr. 1.** A number of men received $120, to be divided equally. If their number had been 4 less, each one would have received three times as much. How many men were there?

Let x stand for the number of men. Then each man received $\frac{120}{x}$ dollars. If their number had been 4 less, each one would have received $\frac{120}{x-4}$ dollars.

Therefore, by the condition of the problem, we have

$$\frac{120}{x-4}=3\times\frac{120}{x};\text{ whence } x=6.$$

Pr. 2. The value of a fraction when reduced to its lowest terms is $\frac{4}{5}$. If its numerator and denominator be each diminished by 1, the resulting fraction will be equal to $\frac{3}{4}$. What is the fraction?

The numerator of the required fraction must be a multiple of 4, and the denominator the same multiple of 5;

Let x stand for this multiple. [illegible]

By the condition of the problem, we have

$$\frac{x-1}{5x-1}=\frac{1}{6}.$$

Whence $x=5$. The required fraction is $\frac{5}{25}$, $=\frac{1}{5}$.

EXERCISES II.

1. What number added to the numerator and denominator of $\frac{3}{7}$ will give a fraction equal to $\frac{3}{4}$?

2. The sum of two numbers is 18, and the quotient of the less divided by the greater is equal to $\frac{1}{5}$. What are the numbers?

3. The denominator of a fraction exceeds its numerator by 2, and if 1 be added to both numerator and denominator, the resulting fraction will be equal to $\frac{2}{3}$. What is the fraction?

4. The sum of a number and seven times its reciprocal is 8. What is the number?

5. The value of a fraction, when reduced to its lowest terms, is $\frac{3}{4}$. If its numerator be increased by 7 and its denominator be decreased by 7, the resulting fraction will be equal to $\frac{4}{3}$. What is the number?

6. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{7}{13}$, to give its reciprocal?

7. If $\frac{1}{4}$ be divided by a certain number increased by $\frac{1}{4}$, and $\frac{1}{4}$ be subtracted from the quotient, the remainder will be $\frac{1}{4}$. What is the number?

8. A train runs 200 miles in a certain time. If it were to run 5 miles an hour faster, it would run 40 miles further in the same time. What is the rate of the train?

9. A number has three digits, which increase by 1 from left to right. The quotient of the number divided by the sum of the digits is 26. What is the number?

10. A number of men have $72 to divide. If $144 were divided among 3 more men, each one would receive $4 more. How many men are there?

11. It was intended to divide $\frac{1}{2}$ by a certain number, but by mistake $\frac{1}{2}$ was added to the number. The result was, nevertheless, the same. What is the number?

12. A steamer can run 20 miles an hour in still water. If it can run 72 miles with the current in the same time that it can run 48 miles against the current, what is the speed of the current?

13. A man buys two kinds of wine, 14 bottles in all, paying $9 for one kind and $12 for the other. If the price of each kind is the same, how many bottles of each does he buy?

14. A can do a piece of work in 10 days, B in 6 days; and A, B, and C together in 3 days. In how many days can C do the work?

15. A and B together can do a piece of work in 2 days, B and C together in 3 days, and A and C together in $2\frac{1}{2}$ days. In how many days can A, B, and C together do the work?

16. The circumference of the hind wheel of a carriage exceeds the circumference of the front wheel by 4 feet, and the front wheel makes the same number of revolutions in running 400 yards that the hind wheel makes in running 500 yards. What is the circumference of each wheel?

17. In a number of two digits, the digit in the tens' place exceeds the digit in the units' place by 2. If the digits be interchanged and the resulting number be divided by the original number, the quotient will be equal to $\frac{23}{32}$. What is the number?

18. In a number of three digits, the digit in the hundreds' place is 2; if this digit be transferred to the units' place, and the resulting number be divided by the original number, the quotient will be equal to $\frac{77}{41}$. What is the number?

19. In one hour a train runs 10 miles further than a man rides on a bicycle in the same time. If it takes the train 6 hours longer to run 255 miles than it takes the man to ride 63 miles, what is the rate of the train?

20. Two engines are used in different places in a mine to pump out water. The one pumps 11 gallons every 5 minutes from a depth of 155 yards; the other pumps 31 gallons every 10 minutes from a depth of 88 yards. The engines together represent the power of 54 horses. Each engine represents the power of how many horses?

21. A cistern has three pipes. To fill it, the first pipe takes one-half of the time required by the second, and the second takes two-thirds of the time required by the third. If the three pipes be open together, the cistern will be filled in 6 hours. In what time will each pipe fill the cistern?

22. A and B ride 100 miles from *P* to *Q*. They ride together at a uniform rate until they are within 30 miles of *Q*, when A increases his rate by $\frac{1}{5}$ of his previous rate. When B is within 20 miles of *Q*, he increases his rate by $\frac{1}{2}$ of his previous rate, and arrives at *Q* 10 minutes earlier than A. At what rate did A and B first ride?

23. A circular road has three stations, *A*, *B*, and *C*, so placed that *A* is 15 miles from *B*, *B* is 13 miles from *C* in the same direction, and *C* is 14 miles from *A* in the same direction. Two messengers leaving *A* at the same time, and traveling in opposite directions, meet at *B*. The faster messenger then reaches *A*, 7 hours before the slower one. What is the rate of each messenger?

CHAPTER XI.

LITERAL EQUATIONS IN ONE UNKNOWN NUMBER.

1. The unknown numbers of an equation are frequently to be determined in terms of general numbers, *i.e.*, in terms of numbers represented by letters. The latter are commonly represented by the leading letters of the alphabet, a, b, c, etc.

Such numbers as a, b, c, etc., are to be regarded as known.

E.g., in the equation $x+a=b$, a and b are the *known* numbers, and x is the *unknown* number.

From this equation we obtain $x=b-a$.

2. It is important to notice that the assumption that x, y, z, etc., are the unknown numbers of an equation, and that a, b, c, etc., are the known numbers, is arbitrary.

In the equation $x+a=b$, either a or b could be taken as the unknown number. If a be taken as the unknown number, we have $a=b-x$; if b be taken as the unknown number, we have $b=x+a$.

3. A **Numerical Equation** is one in which all the known numbers are numerals; as $2x+3=7$; $4x-3y=7$.

A **Literal Equation** is one in which some or all of the known numbers are literal; as $2ax+3b=5$; $ax+by=c$.

4. Ex. 1. Solve the equation $\dfrac{x-a}{b}+\dfrac{x-b}{a}=-\dfrac{(a-b)^2}{2ab}$.

Clearing of fractions,

$$2ax-2a^2+2bx-2b^2=-a^2+2ab-b^2.$$

Transferring and uniting terms,

$$2(a+b)x=a^2+2ab+b^2.$$

Dividing by $2(a+b)$, $\quad x=\dfrac{a+b}{2}$.

Notice that the above equation, although algebraically fractional, is integral in the unknown number x. The equations which follow are fractional in the unknown number.

Ex. 2. Solve the equation $\frac{1-ax}{bx}=\frac{bx-1}{ax}+2.$

Clearing of fractions, $a-a^2x=b^2x-b+2abx.$

Transferring and uniting terms,

$$(a^2+2ab+b^2)x=a+b.$$

Dividing by $a^2+2ab+b^2$, $x=\frac{1}{a+b}.$

Ex. 3. Solve the equation

$$\frac{1}{a-b}+\frac{c}{(x-b)(x-c)}=\frac{x}{(x-b)(x-c)}.$$

Uniting fractions with common denominator,

$$\frac{1}{a-b}+\frac{c-x}{(x-b)(x-c)}=0.$$

Reducing to lowest terms, $\frac{1}{a-b}-\frac{1}{x-b}=0.$

Clearing of fractions, $x-b-a+b=0.$

Whence $x=a.$

Had we cleared of fractions at once, we should have introduced the root c of the factor $x-c$ equated to 0.

5. *A linear equation in one unknown number has one, and only one, distinct root.*

Any linear equation in one unknown number can be reduced to the form

$$ax=b.$$

If both members of this equation be divided by a, when $a\neq 0$, we obtain $x=\frac{b}{a}.$

Since this value of x satisfies the equation, we conclude that every linear equation has at least one root.

Let us assume that the equation $ax=b$ has two distinct roots, and let us denote them by r_1 and r_2. Then, since they must both satisfy the given equation, we have

$$ar_1=b\ (1),\text{ and } ar_2=b\ (2).$$

Subtracting (2) from (1), we obtain

$$a(r_1-r_2)=0.$$

Since $a \neq 0$, we have, by Ch. III., § 3, Art. 18,

$$r_1 - r_2 = 0; \text{ whence } r_1 = r_2.$$

Therefore the assumption that the equation has two distinct roots is untenable. Hence the truth of the principle enunciated.

6. Observe that, in Art. 5, we have no authority for dividing both members of the equation

$$ax = b$$

by a, when $a = 0$. But if we assume that $\frac{b}{a}$ still gives the solution of the equation when $a = 0$, the value of x will be indeterminate ($\frac{0}{0}$) or infinite (∞), according as $b = 0$ or $b \neq 0$. Evidently, when $a = 0$ and $b = 0$, any finite value of x will satisfy the equation; while, when $a = 0$ and $b \neq 0$, no finite value of x will satisfy the equation.

EXERCISES I.

Solve the following equations:

1. $a - x = c$. 2. $mx + a = b$. 3. $a - bx = c$.

4. $mx = nx + 2$. 5. $x - ax + 1 = bx$. 6. $ax + bx - x = 0$.

7. $3ax - 5ab + 6ax - 7ac = 2ax + 2ab$.

8. $4a^2 - 2abx + b^2 + 3a^2x = 5a^2 - b^2x + 2a^2x$.

9. $(2a - b)x = 4a^2 - 3a(b + x)$.

10. $a(x + a) - b(x - b) = 3ax + (a - b)^2$.

11. $x(x + a) + x(x + b) - 2(x + a)(x + b) = 0$.

12. $(a + x)(b + x) - (c - x)(d - x) = 0$.

13. $(3a - x)(a - b) + 2ax = 4b(a + x)$.

14. $a + \frac{b}{x} = c$. 15. $\frac{a}{c} + b = \frac{b}{x} + a$. 16. $\frac{x + a}{x - a} = \frac{5}{4}$.

17. $\frac{b^2}{ax} + \frac{b}{a} - \frac{a}{b} = \frac{a}{x}$. 18. $\frac{a + x}{b + x} = \frac{a + 1}{b + 1}$. 19. $\frac{x+a}{2} - \frac{2}{x+a} = \frac{x-a}{2}$.

20. $\frac{6x+a}{4x+b} - \frac{3x-b}{2x-a} = 0$. 21. $\frac{a + x}{b + a} = \frac{a - x}{b - a}$. 22. $\frac{x - a}{x - 2} = \frac{x + b}{x - 3}$.

23. $\frac{a}{b} = \frac{x - b^2}{x - a^2}$. 24. $\frac{x+ab}{x-ab} = \frac{a^2+ab+b^2}{a^2-ab+b^2}$. 25. $\frac{x + a}{x - b} = \frac{(2x + a)^2}{(2x - b)^2}$.

26. $\frac{x^2 + a^2}{4x^2 - a^2} - \frac{x}{2x + a} = -\frac{1}{4}$. 27. $\frac{a(x + 1) - b(x - 1)}{b(x + 1) - a(x - 1)} = \frac{a^3}{b^3}$.

28. $\frac{a^3 - b^3}{a^3 + b^3} = \frac{a(x - b^2) + b(a^2 - x)}{a(x - b^2) - b(a^2 - x)}$. 29. $\frac{1}{ab - ax} + \frac{1}{bc - bx} - \frac{1}{ac - ax} = 0$.

30. $\frac{x - a}{2bc} + \frac{x - b}{2ac} + \frac{x - c}{2ab} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. 31. $\frac{a - b}{x - 1} + \frac{b - c}{x - 2} + \frac{c - a}{x - 3} = 0$.

32. $\frac{x-a}{b+c}+\frac{x-b}{a+c}+\frac{x-c}{a+b}=\frac{3x}{a+b+c}$.

33. $\frac{a+x}{a^2+ax+x^2}-\frac{a-x}{a^2-ax+x^2}=\frac{3a}{x(a^4+a^2x^2+x^4)}$.

34. $\frac{1-2ax^2}{1+2bx^2}-\frac{1+2ax^2}{1-2bx^2}=\frac{4a^2x^3}{4b^2x^4-1}$.

35. $\frac{a^2+4a}{x^2+x-a^2+a}-\frac{a}{x+a}=\frac{1}{x-a+1}$.

36. $\frac{x-1}{a-1}+\frac{2a^2(1-x)}{a^4-1}=\frac{2x-1}{1-a^4}-\frac{1-x}{1+a}$.

37. $\frac{a^2+x}{b^2-x}-\frac{a^2-x}{b^2+x}=\frac{4abx+2a^2-2b^2}{b^4-x^2}$.

38. $\frac{a^2+ax+x^2}{a^3+a^2x+ax^2+x^3}-\frac{a^3-a^2x+ax^2}{a^4+2a^2x^2+x^4}=\frac{1}{a+x}$.

39. $\frac{a^2-x}{x-2a}-\frac{2a+x}{a^2-x}=\frac{a^4}{a^2x+2ax-2a^3-x^2}$.

40. $\frac{2(x-a)}{a^2-c^2-2ax+x^2}+\frac{c-x}{a^2-ac+cx-2ax+x^2}=\frac{1}{x-(a+c)}$.

41. $1-\frac{1-\frac{1}{a^2}}{\frac{a}{x}\left(1-\frac{1}{a}\right)}=-\frac{1}{a^3}$. 42. $\frac{a+x-\frac{a^2}{a+x}}{a+x}=1-\frac{2ax}{(a+x)^2}$.

General Problems.

7. A problem in which the given numbers have particular values can be *generalized* by assuming literal numbers for the given numbers. A problem so stated is a *general problem*, and its solution is called a *general solution*.

A *particular* solution can be obtained from the general solution by assigning to the literal numbers in the latter, particular numerical values.

We will now obtain general solutions of some of the problems already solved in Ch. V.

Ch. V., Pr. 1. The greater of two numbers is m times the less, and their sum is s. What are the numbers?

Let x stand for the less required number. Then mx stands for the greater. By the condition of the problem, we have

$$x+mx=s;$$

whence, $x=\frac{s}{1+m}$, the less number, and $mx=\frac{ms}{1+m}$, the greater.

If $m = 3$ and $s = 84$ (as in the particular problem), we have

$$x = \frac{84}{1+3} = 21, \text{ and } mx = 3 \times 21 = 63.$$

When the numbers are equal, $m = 1$, and we obtain

$$x = \frac{s}{2} \text{ and } mx = \frac{s}{2},$$

for all values of s; that is, either of the two numbers is half their sum.

Ch. V., Pr. **9**. A carriage, starting from a point A, travels m miles daily; a second carriage, starting from a point B, p miles behind A, travels in the same direction n miles daily. After how many days will the second carriage overtake the first, and at what distance from B will the meeting take place?

Let x stand for the number of days after which the carriages meet. Then the number of miles traveled by the first carriage will be mx; the number traveled by the second will be nx.

Therefore, by the condition of the problem,

$$mx = nx - p; \text{ whence } x = \frac{p}{n-m},$$

the number of days after which the carriages meet.

The distance traveled by the first carriage is $\frac{mp}{n-m}$ miles, and the distance traveled by the second carriage is $\frac{np}{n-m}$ miles.

They, therefore, meet $\frac{np}{n-m}$ miles from B.

Ch. V., Pr. **10**. One man asked another what time it was, and received the answer: "It is between n and $n+1$ o'clock, and the minute-hand is directly over the hour-hand." What time was it?

At n o'clock the minute-hand points to 12 and the hour-hand to n. The hour-hand is therefore $5n$ minute-divisions in advance of the minute-hand.

Let x stand for the number of minute-divisions passed over by the minute-hand from n o'clock until it is directly over the hour-hand between n and $n+1$ o'clock.

Then the number of minute-divisions passed over by the our-hand is equal to the number of minute-divisions passed ver by the minute-hand, minus $5n$; that is, to $x - 5n$.

But since the minute-hand moves 12 times as fast as the our-hand, we have

$$x = 12(x - 5n); \text{ whence } x = \frac{60n}{11}.$$

Consequently, the time was $\frac{60n}{11}$ minutes past n.

If $n = 1$, it was $\frac{60}{11}$, or $5\frac{5}{11}$ minutes past 1.

If $n = 2$, it was $\frac{120}{11}$, or $10\frac{10}{11}$ minutes past 2.

Etc.

If $n = 11$, it was $\frac{660}{11}$, or 60 minutes past 11; *i.e.*, 12 o'clock.

If $n = 12$, it was $\frac{720}{11}$, or $65\frac{5}{11}$ minutes past 12; *i.e.*, $5\frac{5}{11}$ minutes past 1.

Notice that the two hands coincide at 12 o'clock, but not between 12 and 1.

EXERCISES II.

Find the general solution of each of the following problems, and from this solution obtain the particular solution for the numerical values assigned to the literal numbers in the problem.

1. Find a number, such that the result of adding it to n shall be equal to n times the number. Let $n = 2$; 5.

2. Divide a into two parts, such that $\frac{1}{m}$ of the first, plus $\frac{1}{n}$ of the second, shall be equal to b. Let $a = 100$, $b = 30$, $m = 3$, $n = 5$.

3. Find a number, such that the sum of the results of subtracting it from a and from b shall be equal to c. Let $a = 3$, $b = 6$, $c = 5$.

4. One boy said to another: "Think of some number, add 3 to it, multiply the sum by 2, add 4 to the product, divide the result by 2, subtract 1 from the quotient, multiply the difference by 4, add 4 to the product, divide the result by 4, tell me the result, and I will tell you the number you have in mind." If the result be d, what number does the boy think of?

5. A sum of d dollars is divided between A and B. B receives b dollars as often as A receives a dollars. How much does each receive? Let $d = 7000$, $a = 3$, $b = 2$.

6. A father's age exceeds his son's age by m years, and the sum of their ages is n times the son's age. What are their ages? Let $m = 20$, $n = 4$; $m = 25$, $n = 7$.

7. If two trains start together and run in the same direction, one at the rate of m_1 miles an hour, and the other at the rate of m_2 miles an hour, after how many hours will they be d miles apart? Let $d = 200$, $m_1 = 35$, $m_2 = 30$.

8. A farmer can plow a field in a days, and his son in b days; in how many days can they plow the field, working together? Let $a = 10$, $b = 15$.

9. A pupil was told to add m to a certain number, and to divide the sum by n. But he misunderstood the problem, and subtracted n from the number and multiplied the remainder by m. Nevertheless he obtained the correct result. What was the number? Let $m = 12$, $n = 13$.

10. What time is it, if the number of hours which have elapsed since noon is m times the number of hours to midnight? Let $m = \frac{1}{2}$.

11. A starts from P and walks to Q, a distance of d miles. At the same time B starts from Q and walks to P. If A walk at the rate of m miles a day and B at the rate of n miles a day, at what distance from P do they meet, and how many days after they start? Let $m = 20$, $n = 30$, $d = 600$.

12. Two friends, A and B, each intending to visit the other, start from their houses at the same time. A could reach B's house in m minutes, and B could reach A's house in n minutes. After how many minutes do they meet? Let $m = 12\frac{1}{4}$, $n = 10\frac{1}{2}$.

13. Two couriers start at the same time and move in the same direction, the first from a place d miles ahead of the second. The first courier travels at the rate of m_1 miles an hour, and the second at the rate of m_2 miles an hour. After how many hours will the second courier overtake the first? Let $d = 15$, $m_1 = 17$, $m_2 = 20$.

From the result of the preceding example find the results of Exx. 14–16.

14. At what rate must the second courier travel in order to overtake the first after h hours? Let $d = 18$, $m_1 = 15$, $h = 3$.

15. At what rate must the first courier travel in order that the *second* may overtake him after h hours? Let $d = 12$, $m_2 = 22$, $h = 3$.

16. How many miles behind the first courier must the second start in order to overtake the first after h hours? Let $m_1 = 18$, $m_2 = 21$, $h = 4$.

17. In a company are a men and b women; and to every m unmarried men there are n unmarried women. How many married couples are in the company? Let $a = 13$, $b = 17$, $m = 3$, $n = 5$.

18. The annual dues of a certain club are at first a dollars. Subsequently the yearly expenses increased by d dollars, while the number of members decreased by n. In consequence the annual dues were increased by b dollars. How many members were originally in the club? Let $a = 25$, $d = 315$, $n = 7$, and $b = 2$.

19. A father divided his property equally among his sons. To the oldest he gave d dollars and $\frac{1}{n}$ of what remained; to the second son he gave $2d$ dollars and $\frac{1}{n}$ of what was then left; to the third son he gave $3d$ dollars and $\frac{1}{n}$ of the remainder; and so on. What was the amount of his property? Let $d = 1500$, $n = 11$; $d = 2000$, $n = 6$..

20. Two couriers start from the same place and move in the same direction, one h hours after the other. The first one travels at the rate of m_1 miles an hour, and the second at the rate of m_2 miles an hour. After how many hours will the second courier overtake the first? Let $h = 2$, $m_1 = 15$, $m_2 = 20$.

From the result of the preceding example, find the results of Exx. 21–23.

21. At what rate must the second courier travel in order to overtake the first after H hours? Let $H = 6$, $h = 2$, $m_1 = 12$.

22. At what rate must the first courier travel in order that the second may overtake him after H hours? Let $H = 4$, $h = 1$, $m_2 = 20$.

23. How many hours after the first courier starts must the second start in order to overtake the first after H hours? Let $H = 6$, $m_1 = 14$, $m_2 = 22$.

24. Two boys run a race from A to B, a distance of d yards. The first runs a yards a second; after reaching B, he turns and runs back at the same rate to meet the other boy, who runs b yards a second. How many seconds after they start does the faster runner meet the other? Let $d = 253$, $a = 2.5$, $b = 2.1$.

25. An accommodation train leaves A every h hours, and runs to B at the rate of m miles an hour. At the same time an express train leaves B and runs to A at the rate of n miles an hour. What time elapses after an express train meets an accommodation train until it meets the next accommodation train? Let $h = 3$, $m = 20$, $n = 40$.

26. At what time between n and $n + 1$ o'clock will the hands of a clock be in a straight line? Let $n = 1$; 2; 3; $\cdots$ to 12.

27. At what time between n and $n + 1$ o'clock are the minute-hand and the hour-hand of a clock at right angles to each other? Let $n = 1$; 2; 3; $\cdots$ to 12.

28. At what time between n and $n + 1$ o'clock does the second-hand bisect the angle between the hour-hand and the minute-hand? Let $n = 1$; 2; 3; $\cdots$ to 12.

CHAPTER XII.

INTERPRETATION OF THE SOLUTIONS OF PROBLEMS.

1. In solving equations we do not concern ourselves with the meaning of the results. When, however, an equation has arisen in connection with a problem, the interpretation of the result becomes important. In this chapter we shall interpret the solutions of some linear equations in connection with the problems from which they arise.

Positive Solutions.

2. Pr. A company of 20 people, men and women, proposed to arrange a fair for the benefit of a poor family. Each man contributed $3, and each woman $1. If $55 were contributed, how many men and how many women were in the company?

Let x stand for the number of men; then the number of women was $20 - x$. The amount contributed by the men was $3x$ dollars, that by the women $20 - x$ dollars. By the condition of the problem, we have

$$3x + (20 - x) = 55; \text{ whence } x = 17\tfrac{1}{2}.$$

The result, $17\frac{1}{2}$, satisfies the equation, but not the problem. For the number of men must be an *integer*. This implied condition could not be introduced into the equation.

The conditions stated in the problem are impossible, since they are inconsistent with the implied condition.

If the problem be generalized, its solution will show how the given data can be modified so that all the conditions, expressed and implied, shall be consistent. The generalized problem may be stated thus:

A company of m people, men and women, proposed to arrange a fair for the benefit of a poor family. Each man contributed a dollars, and each woman b dollars. If n dollars were contributed, how many men and how many women were in the company?

The solution of the equation of this problem is

$$x = \frac{n - bm}{a - b}.$$

n order that x may be an integer, $n - bm$ must be exactly divisible by $-b$. Thus, if, in the given problem, the number of people were 21 nstead of 20, the other data being the same, we should have

$$x = \frac{55 - 1 \times 21}{3 - 1} = \frac{34}{2} = 17.$$

If all the conditions of a problem, expressed and implied, be consistent, positive solution will satisfy these conditions and therefore give the solution of the problem.

Negative Solutions.

3. Pr. A father is 40 years old, and his son 10 years old. After how many years will the father be seven times as old as his son?

Let x stand for the required number of years. Then after x years the father will be $40 + x$ years old, and the son $10 + x$ years old. By the condition of the problem, we have

$$40 + x = 7(10 + x), \text{ whence } x = -5. \tag{1}$$

This result satisfies the equation, but not the condition of the problem. For since the question of the problem is "*after* how many years?" the result, if added to the number of years in the ages of father and son, should increase them, and therefore be *positive*. Consequently, at no time in the future will the father be seven times as old as his son. But since to add -5 is equivalent to subtracting 5, we conclude that the question of the problem should have been, "How many years ago?"

The equation of the problem, with this modified question, is:

$$40 - x = 7(10 - x); \text{ whence } x = 5. \tag{2}$$

Notice that equation (2) could have been obtained from equation (1) by changing x into $-x$.

4. The interpretation of a negative result in a given problem is often facilitated by the following principle:

If $-x$ be substituted for x in an equation which has a negative root, the resulting equation will have a positive root of the same absolute value; and vice versa.

E.g., the equation $x + 1 = -x - 3$ has the root -2;
while the equation $-x + 1 = x - 3$ has the root 2.

In general, the equation $ax = b$ has the root $\frac{b}{a}$. (1)

And the equation $-ax = b$ (2)

has the root $-\frac{b}{a}$. If the root $\frac{b}{a}$ be negative, then the root $-\frac{b}{a}$ is positive; and *vice versa*.

5. Pr. Two pocket-books contain together $100. If one-half of the contents of one pocket-book, and one-third of the contents of the other be removed, the amount of money left in both will be $70. How many dollars does each pocket-book contain?

Let x stand for the number of dollars contained in the first pocket-book; then the number of dollars contained in the second is $100-x$. When one-half of the contents of the first, and one-third of the contents of the second are removed, the number of dollars remaining in the first is $\frac{x}{2}$, and in the second $\frac{2}{3}(100-x)$. By the conditions of the problem, we have

$$\tfrac{1}{2}x+\tfrac{2}{3}(100-x)=70, \text{ whence } x=-20.$$

Substituting $-x$ for x in the given equation, we obtain

$$-\tfrac{1}{2}x+\tfrac{2}{3}(100+x)=70, \text{ or } \tfrac{2}{3}(100+x)-\tfrac{1}{2}x=70.$$

This equation corresponds to the following conditions:

If x stand for the number of dollars in one pocket-book, then $100+x$ stands for the number of dollars in the other; that is, one pocket-book contains $100 more than the other. The second condition of the problem, obtained from the equation, is: two-thirds of the contents of one pocket-book exceeds one-half of the contents of the other by $70. Therefore the modified problem reads as follows:

Two pocket-books contain a certain amount of money, and one contains $100 more than the other. If one-third of the contents be removed from the first pocket-book, and one-half of the contents from the second, the first will then contain $70 more than the second. How much money is contained in each pocket-book?

6. These problems show that the required modification of an assumption, question, or condition of a problem which has led to a negative result, consists in making the assumption, question, or condition the opposite of what it originally was.

Thus, if a positive result signify a distance toward the right from a certain point, a negative result will signify a distance toward the left from the same point; and *vice versa;* etc.

Zero Solutions.

7. A zero result gives in some cases the answer to the question; in other cases it proves its impossibility.

Pr. A merchant has two kinds of wine, one worth $7.25 a gallon, and the other $5.50 a gallon. How many gallons of each kind must be taken to make a mixture of 16 gallons worth $88?

Let x stand for the number of gallons of the first kind; then $16-x$ will stand for the number of gallons of the second kind.

Therefore, by the condition of the problem, we have

$$7.25\,x + 5.5\,(16 - x) = 88\,;\ \text{whence}\ x = 0.$$

That is, no mixture which contains the first kind of wine can be made to satisfy the condition. In fact, 16 gallons of the second kind are worth $88.

Indeterminate Solutions.

8. Pr. A merchant buys 4 pieces of goods. In the second piece there are 3 yards less than in the first, in the third 7 yards less than in the first, and in the fourth 10 yards less than in the first. The number of yards in the first and fourth is equal to the number of yards in the second and third. How many yards are there in the first piece?

Let x stand for the number of yards in the first piece; then the number of yards in the second piece is $x - 3$; in the third piece, $x - 7$; in the fourth piece, $x-10$. Therefore, by the condition of the problem, we have

$$x + (x - 10) = (x - 3) + (x - 7),\ \text{ or }\ 2\,x - 10 = 2\,x - 10.$$

This equation is an identity, and is therefore satisfied by any finite value of x.

If it be solved in the usual way, we obtain

$$(2 - 2)x = 10 - 10,\ \text{ or }\ x = \frac{10 - 10}{2 - 2} = \frac{0}{0}.$$

That is, the conditions of the problem will be satisfied by any number of yards in the first piece.

Infinite Solutions.

9. Pr. A cistern has three pipes. Through the first it can be filled in 24 minutes; through the second in 36 minutes; through the third it can be emptied in $14\frac{2}{5}$ minutes. In what time will the cistern be filled if all the pipes be opened at the same time?

Let x stand for the number of minutes after which the cistern will be filled. In one minute $\frac{1}{24}$ of its capacity enters through the first pipe, and hence in x minutes $\dfrac{x}{24}$ of its capacity enters. For a similar reason, $\dfrac{x}{36}$ of its capacity enters through the second pipe in x minutes; and in the same time $\dfrac{5\,x}{72}$ of its capacity is discharged through the third pipe.

Therefore, after x minutes there is in the cistern

$$\frac{x}{24} + \frac{x}{36} - \frac{5\,x}{72},\ = \left(\tfrac{1}{24} + \tfrac{1}{36} - \tfrac{5}{72}\right)x,$$

of its capacity. But by the condition of the problem, that the cistern is then filled, we have

$$\left(\tfrac{1}{24} + \tfrac{1}{36} - \tfrac{5}{72}\right)x = 1\,;$$

whence

$$x = \frac{1}{\frac{1}{24} + \frac{1}{36} - \frac{5}{72}} = \frac{1}{0} = \infty.$$

This result means that the cistern will never be filled. This is also evident from the data of the problem, since the third pipe in a given time discharges from the cistern as much as enters it through the other pipes.

The Problem of the Couriers.

10. Pr. Two couriers are traveling along a road in the direction from M to N; one courier at the rate of m_1 miles an hour, the other at the rate of m_2 miles an hour. The former is seen at the station A at noon, and the other is seen h hours later at the station B, which is d miles from A in the direction in which the couriers are traveling. Where do the couriers meet?

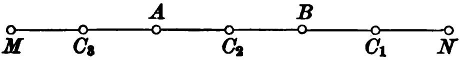


FIG. 3.

Assume that they meet to the right of B at a point C_1, and let x stand for the number of miles from B to the place of meeting C_1 (Fig. 3).

The first courier, moving at the rate of m_1 miles an hour, travels $d+x$ miles, from A to C_1, in $\frac{d+x}{m_1}$ hours; the second courier, moving at the rate of m_2 miles an hour, travels x miles, from B to C_1, in $\frac{x}{m_2}$ hours. By the condition of the problem it is evident that, if the place of meeting be to the right of B, the number of hours it takes the first courier to travel from A to C_1 exceeds by h the number of hours it takes the second courier to travel from B to C_1. We therefore have

$$\frac{d+x}{m_1}-\frac{x}{m_2}=h, \tag{1}$$

whence $$x=\frac{hm_1m_2-dm_2}{m_2-m_1}=\frac{m_2(hm_1-d)}{m_2-m_1}.$$

(i.) **A Positive Result.** — The result will be positive either when $hm_1 > d$ and $m_2 > m_1$, or when $hm_1 < d$ and $m_2 < m_1$. A positive result means that the problem is possible with the assumption made; *i.e.*, that the couriers meet at a point to the right of B.

(ii.) **A Negative Result.** — The result will be negative either when $hm_1 > d$ and $m_2 < m_1$, or when $hm_1 < d$ and $m_2 > m_1$. Such a result shows that the assumption that the couriers meet to the right of B is untenable, since, as we have seen, in that case the result is positive.

That under the assumed conditions the couriers can meet only at some point to the left of B can also be inferred from the following considerations, which are independent of the negative result: If $hm_1 > d$, the first courier has passed B when the second courier is seen at that station; that is, the second courier is behind the first at that time. And since also $m_2 < m_1$, the first courier is traveling the faster, and must therefore have overtaken the second, and at some point to the left of B.

On the other hand, if $hm_1 < d$, the first courier has not yet reached B when the second is seen at that station; that is, the first courier is behind the second at that time. And since also $m_2 > m_1$, the second courier is traveling the faster, and must therefore have overtaken the first, at some point to the left of B. Similar reasoning could have been applied in (i.).

(iii.) **A Zero Result.** — A zero result is obtained when $hm_1 = d$, and $m_2 \neq m_1$; that is, the meeting takes place at B. This is also evident from the assumed conditions. For the first courier reaches B h hours after he was seen at A; and since the second courier is seen at B, h hours after the first was seen at A, the meeting must take place at B.

(iv.) **Indeterminate Result.** — An indeterminate result is obtained if $hm_1 = d$, and $m_2 = m_1$. In this case every point of the road can be regarded as their place of meeting. For the first courier evidently reaches B at the time at which the second courier is seen at that station; and since they are traveling at the same rate, they must be together all the time. The problem under these conditions becomes indeterminate.

(v.) **An Infinite Result.** — An infinite result is obtained when $hm_1 \neq d$, and $m_2 = m_1$. In this case a meeting of the couriers is impossible, since both travel at the same rate, and when the second is seen at B the first either has not yet reached B or has already passed that station.

An infinite result also means that the more nearly equal m_1 and m_2 are, the further removed is the place of meeting.

EXERCISES.

Solve the following problems, and interpret the results. Modify those problems which have negative solutions so that they will be satisfied by positive solutions.

1. A and B together have $100. If A spend one-third of his share, and B spend one-fourth of his share, they will then have $80 left. What are their respective shares?

2. In a number of two digits, the digit in the tens' place exceeds the digit in the units' place by 5. If the digits be interchanged, the resulting number will be less than the original number by 45. What is the number?

3. A father is 40 years old, and his son is 13 years old; after how many years will the father be four times as old as his son?

4. The sum of the first and third of three consecutive even numbers is equal to twice the second. What are the numbers?

5. In a number of two digits, the tens' digit is two-thirds of the units' digit. If the digits be interchanged, the resulting number will exceed the original number by 36. What is the number?

6. A father is 26 years older than his son, and the sum of their ages is 26 years less than twice the father's age. How old is the son?

7. A teacher proposes 30 problems to a pupil. The latter is to receive 8 marks in his favor for each problem solved, and 12 marks against him for each problem not solved. If the number of marks against him exceed those in his favor by 420, how many problems will he have solved?

8. In a number of two digits the tens' digit is twice the units' digit. If the digits be interchanged, the resulting number will exceed the original number by 18. What is the number?

9. In a number of two digits, the digit in the units' place exceeds the digit in the tens' place by 4. If the sum of the digits be divided by 2, the quotient will be less than the first digit by 2. What is the number?

10. A has $100, and B has $30. A spends twice as much money as B, and then has left three times as much as B. How much does each one spend?

Discuss the solutions of the following general problems. State under what conditions each solution is positive, negative, zero, indeterminate, or infinite. Also, in each problem, assign a set of particular values to the general numbers which will give an admissible solution.

11. In a number of two digits, the tens' digit is m times the units' digit. If the digits be interchanged, the resulting number will exceed the original number by n. What is the number?

12. A father is a years old, and his son is b years old. After how many years will the father be n times as old as his son?

13. What number, added to the denominators of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will make the resulting fractions equal?

14. Having two kinds of wine worth a and b dollars a gallon, respectively, how many gallons of each kind must be taken to make a mixture of n gallons worth c dollars a gallon?

15. Two couriers, A and B, start at the same time from two stations, distant d miles from each other, and travel in the same direction. A travels n times as fast as B. Where will A overtake B?

CHAPTER XIII.

SIMULTANEOUS LINEAR EQUATIONS.

§ 1. SYSTEMS OF EQUATIONS.

1. If the linear equation in two unknown numbers

$$x + y = 5 \qquad (1)$$

be solved for y, we obtain

$$y = 5 - x.$$

This value of y is not definitely determined. We may substitute in it any particular numerical value for x, and obtain a corresponding value for y. Thus,

when $x = 1$, $y = 4$; when $x = 2$, $y = 3$; when $x = 3$, $y = 2$; etc.

In like manner the equation could have been solved for x in terms of y, and corresponding sets of values obtained.

Any set of corresponding values of x and y satisfies the given equation, and is therefore a solution.

An equation which, like the above, has an indefinite number of solutions, is called an **Indeterminate Equation.**

2. The equation $\qquad y - x = 1 \qquad (2)$

also has an unlimited number of solutions. Solving this equation for y, we have $y = 1 + x$. Then,

when $x = 1$, $y = 2$; when $x = 2$, $y = 3$; when $x = 3$, $y = 4$; etc.

Now, observe that equations (1) and (2) have the common solution, $x = 2$, $y = 3$. It seems evident, and we shall later prove, that these equations have only this solution in common.

Equations (1) and (2) express different relations between the unknown numbers, and are called **Independent Equations.**

Also, since they are satisfied by a common set of values of the unknown numbers, they are called **Consistent Equations.**

3. The equations $x + y = 5$ and $3x + 3y = 16$ are not satisfied by any common set of values of x and y.

For any set of values which reduces $x + y$ to 5 must reduce $3x + 3y$, or $3(x + y)$, to 15, and not to 16. These two equations express inconsistent relations between the unknown numbers, and are called **Inconsistent Equations.**

4. The three equations

$$x + y = 5 \ (1), \quad y - x = 1 \ (2), \quad 2x + y = 9 \ (3),$$

are not satisfied by any common set of values of x and y. For, by Art. 2, equations (1) and (2) are satisfied by the values $x = 2$, $y = 3$. But equation (3) is evidently not satisfied by this set of values. The three equations express three independent relations between x and y.

5. A **System of Simultaneous Equations** is a group of equations which are to be satisfied by the same set, or sets, of values of the unknown numbers.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which converts all of the equations into identities; that is, which satisfies all of the equations.

The examples of Arts. 1–4 are illustrations of the following general principles, which will be proved later:

A system of linear equations has a definite number of solutions,

(i.) *When the number of equations is the same as the number of unknown numbers.*

(ii.) *When the equations are independent and consistent.*

EXERCISES I.

1. Are equivalent equations consistent? Are they independent?

Which of the following systems have inconsistent equations? Which have equations not independent? Which have equations consistent and independent?

2. $\begin{cases} 3x + 5y = 11, \\ 4x + 7y = 15. \end{cases}$ **3.** $\begin{cases} 2x + 3y = 4, \\ 4x - 6y = 8. \end{cases}$ **4.** $\begin{cases} 3x + 10y = 42, \\ 6x + 20y = 84. \end{cases}$

5. $\begin{cases} 6x - 9y = 4, \\ 4x - 6y = 9. \end{cases}$ **6.** $\begin{cases} 18x - 15y = 51, \\ 6x - 5y = 17. \end{cases}$ **7.** $\begin{cases} 5x + 4y = 6, \\ 7x + 6y = 10. \end{cases}$

§ 2. EQUIVALENT SYSTEMS.

1. *Two systems of equations are equivalent when every solution of either system is a solution of the other.*

E.g., the systems (I.) and (II.):

$$\left.\begin{aligned} 3x + 2y &= 8, \\ x - y &= 1, \end{aligned}\right\} \quad \text{(I.)} \qquad \left.\begin{aligned} 3x + 2y &= 8, \\ 2x - 2y &= 2, \end{aligned}\right\} \quad \text{(II.)}$$

are equivalent. For they are both satisfied by the solution, $x = 2$, $y = 1$, and, as we shall see later, by no other solution.

2. The solution of a system of equations depends also upon the following principles of the equivalence of systems:

(i.) *If any equation of a system be replaced by an equivalent equation, the resulting system will be equivalent to the given one.*

Thus, the systems (I.) and (II.) above are equivalent. The equation $x - y = 1$ of (I.) is replaced by the equivalent equation $2x - 2y = 2$.

(ii.) *If any equation of a system be replaced by an equation obtained by adding or subtracting corresponding members of two or more of the equations of the system, the resulting system will be equivalent to the given one.*

Thus, the system (II.) is equivalent to the system

$$\left.\begin{aligned} 3x + 2y &= 8, \\ (3x + 2y) + (2x - 2y) &= 10, \end{aligned}\right\} \text{ or } \left.\begin{aligned} 3x + 2y &= 8, \\ 5x &= 10. \end{aligned}\right\} \quad \text{(III.)}$$

(iii.) *If one equation of a system be solved for one of the unknown numbers, and the resulting value be substituted for this unknown number in each of the other equations, the derived system will be equivalent to the given one.*

E.g., the systems

$$\left.\begin{aligned} x - y &= 2, \\ 5x - 3y &= 12, \end{aligned}\right\} \quad \text{(IV.)} \quad \text{and} \quad \left.\begin{aligned} x &= 2 + y, \\ 5(2 + y) - 3y &= 12, \end{aligned}\right\} \quad \text{(V.)}$$

are equivalent.

The proofs of the principles enunciated are as follows:

(i.) Let $A = B$, and $C = D$, (I.)

be two equations in two unknown numbers, say x and y; and let $C' = D$ be equivalent to $C = D$.

Then the system $A = B$, and $C' = D'$, (II.)

is equivalent to the system (I.). For, by definition of equivalent equations, the same sets of values which satisfy $C = D$ also satisfy $C' = D'$, and *vice versa*. Therefore, any one of these sets of values, which also satisfies $A = B$, is a solution of both systems. Consequently, every solution of either system is a solution of the other.

In like manner, the principle can be proved for a system of any number of equations.

(ii.) The proof is very similar to that of Ch. IV., § 3, Art. 5 (i.).

(iii.) Let $A = B$, (1) and $C = D$, (2) (I.)

be two equations in two unknown numbers, say x and y; and let $x = P$ be the equation derived by solving (1) for x, and $C' = D'$ be the equation obtained by substituting P for x in (2).

Then the system $x = P$, (3) and $C' = D'$, (4) (II.)

is equivalent to the system (I.).

Since equation (3) is equivalent to equation (1), any solution of the system (1) must satisfy equation (3); that is, must give to x and P one and the same value. But (4) differs from (2) only in having P where (2) has x. Therefore, since x and P have the same value, any value of x, with the corresponding value of y, which makes C and D equal must make C' and D' equal. Therefore, every solution of the system (I.) is a solution of the system (II.).

Since equation (1) is equivalent to equation (3), any solution of the system (II.) must satisfy equation (1); that is, must make A and B equal. But (2) differs from (4) only in having x where (4) has P. Therefore, since any solution of (II.) makes x and P equal and C' and D' equal, it must also make C and D equal. Therefore, every solution of the system (II.) is a solution of the system (I.).

Consequently, the two systems are equivalent.

In like manner, the principle can be proved for a system of any number of equations.

3. Elimination is the process of deriving from two or more equations of a system an equation with one less unknown number than the equations from which it is derived. The unknown number which does not appear in the derived equation is said to have been *eliminated*.

E.g., if the equations

$$x + y = 7,$$
$$x - y = 1,$$

be added, we obtain

$$2x = 8,$$

in which the unknown number y does not appear. We say that y has been eliminated from the given equations.

§ 3. SYSTEMS OF LINEAR EQUATIONS.

Linear Equations in Two Unknown Numbers.

1. There are several methods for solving two simultaneous equations in two unknown numbers. The object in all of them is to obtain from the given system an equivalent system of which one equation contains only one unknown number.

Elimination by Addition and Subtraction.

2. Ex. 1. Solve the system

$$\left.\begin{aligned} 3x + 4y &= 24, \quad (1)\\ 5x - 6y &= 2. \quad (2)\end{aligned}\right\} \qquad \text{(I.)}$$

To eliminate x, we multiply both members of equation (1) by 5, and both members of equation (2) by 3, thereby making the coefficients of x in the two equations equal. We then have

$$\left.\begin{aligned} 15x + 20y &= 120, \quad (3)\\ 15x - 18y &= 6. \quad (4)\end{aligned}\right\} \qquad \text{(II.)}$$

The system (II.) is equivalent to the system (I.), by § 2, Art. 2 (i.). The system (II.) is, by § 2, Art. 2 (i.) and (ii.), equivalent to the system

$$\left.\begin{aligned} 3x + 4y &= 24, \quad (1)\\ (15x + 20y) - (15x - 18y) &= 120 - 6; \quad (5)\end{aligned}\right\} \qquad \text{(III.)}$$

or, performing the indicated operations, to

$$\left.\begin{aligned} 3x + 4y &= 24, \quad (1)\\ 38y &= 114; \quad (6)\end{aligned}\right\} \text{(IV.)} \quad \text{or} \quad \left.\begin{aligned} 3x + 4y &= 24, \quad (1)\\ y &= 3. \quad (7)\end{aligned}\right\} \text{(V.)}$$

The system (V.) gives the required solution, since equation (7) gives the value of y, and equation (1) the corresponding value of x, by § 2, Art. 2 (iii.). Substituting 3 for y in (1), we obtain

$$3x + 12 = 24; \text{ whence } x = 4.$$

Consequently the required solution is $x = 4$, $y = 3$.

This solution may be written 4, 3, it being understood that the first number is the value of x, and the second the value of y.

The work has been given in full to emphasize that by each step one system has been replaced by an equivalent system. In practice the work may be contracted as follows:

Multiplying (1) by 5, $15x + 20y = 120.$ (3)

Multiplying (2) by 3, $15x - 18y = 6.$ (4)

Subtracting (4) from (3), $38y = 114;$ (6)

whence $y = 3.$ (7)

Substituting 3 for y in (1), $3x + 12 = 24;$ (8)

whence $x = 4.$ (9)

In a similar way y could have been first eliminated.

Ex. 2. Solve the system

$$\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5, \tag{1}$$

$$\frac{4x-3}{6} + \frac{5y-7}{2} = 18 - 5x. \tag{2}$$

Clearing (1) and (2) of fractions,

$$28 + 4x - 10x + 5y = 60y - 100, \tag{3}$$

$$4x - 3 + 15y - 21 = 108 - 30x. \tag{4}$$

Transferring and uniting terms,

$$6x + 55y = 128, \tag{5}$$

$$34x + 15y = 132. \tag{6}$$

Multiplying (5) by 3, $18x + 165y = 384.$ (7)

Multiplying (6) by 11, $374x + 165y = 1452.$ (8)

Subtracting (7) from (8), $356x = 1068;$ (9)

whence $x = 3.$ (10)

Substituting 3 for x in (5), $18 + 55y = 128$; (11)

whence $y = 2$. (12)

Consequently, the required solution is 3, 2.

Notice that (1) and (2), (3) and (4), (5) and (6), (7) and (8), and (10) and any preceding equation except (9), form equivalent systems. In forming with (10) an equivalent system, we take the simplest of the preceding equations, in this case (5).

3. The examples of the preceding article illustrate the following method of elimination by addition and subtraction.

Simplify the given equations, if necessary, and transfer the terms in x and y to the first members, and the terms free from x and y to the second members.

Determine the L. C. M. of the coefficients of the unknown number to be eliminated, and multiply both members of each equation by the quotient of the L. C. M. divided by the coefficient of that unknown number in the equation.

The coefficients of the unknown number to be eliminated being now equal, or equal and opposite, in the two equations, subtract, or add, corresponding members, and equate the results. A final equation in one unknown number will thus be derived.

The solution of the given system is then obtained by solving this derived equation, and substituting the value of the unknown number thus obtained in the simplest of the preceding equations.

EXERCISES II.

Solve the following systems of equations by the method of addition and subtraction:

1. $\begin{cases} x + y = 17, \\ x - y = 7. \end{cases}$ **2.** $\begin{cases} x + y = a, \\ x - y = b. \end{cases}$ **3.** $\begin{cases} 7x + 11y = 2, \\ 7x - 11y = 0. \end{cases}$

4. $\begin{cases} x - 12y = 3, \\ x + 4y = 19. \end{cases}$ **5.** $\begin{cases} 3x + y = 31, \\ 5x - 2y = 15. \end{cases}$ **6.** $\begin{cases} 4x - 7y = 19, \\ x + 9y = 37. \end{cases}$

7. $\begin{cases} 10x - 3y = 25, \\ 5x - 9y = -25. \end{cases}$ **8.** $\begin{cases} nx - ay = 0, \\ n^2x - ay = an. \end{cases}$ **9.** $\begin{cases} 6x + y = 6, \\ 4x + 3y = 11. \end{cases}$

10. $\begin{cases} 12x + 15y = 8, \\ 16x + 9y = 7. \end{cases}$ **11.** $\begin{cases} 5x + 4y = 49\frac{1}{2}, \\ 2x + 7y = 63. \end{cases}$ **12.** $\begin{cases} 5x - 3y = 12, \\ 19x - 5y = 73\frac{1}{3}. \end{cases}$

13. $\begin{cases} 3x+16y=5, \\ -5x+28y=19. \end{cases}$

14. $\begin{cases} 21x+8y=-66, \\ 28x-23y=13. \end{cases}$

15. $\begin{cases} 18x-20y=1, \\ 15x+16y=9. \end{cases}$

16. $\begin{cases} 12x-14y=-4, \\ 8x-21y=-8.5. \end{cases}$

17. $\begin{cases} \dfrac{2x-y}{2}+14=18, \\ \dfrac{2y+x}{3}+16=19. \end{cases}$

18. $\begin{cases} 3x+\dfrac{7y}{2}=22, \\ 11y-\dfrac{2x}{5}=20. \end{cases}$

19. $\begin{cases} \dfrac{x}{m-a}+\dfrac{y}{m-b}=1, \\ \dfrac{x}{n-a}+\dfrac{y}{n-b}=1. \end{cases}$

20. $\begin{cases} \dfrac{ax-by}{a^2+b^2}=1, \\ \dfrac{bx+ay}{a^2+b^2}=1. \end{cases}$

Elimination by Substitution.

4. Ex. Solve the system

$$\left.\begin{aligned} 5x-2y&=1, && (1)\\ 4x+5y&=47. && (2)\end{aligned}\right\}\ (\text{I.})$$

If we wish to eliminate x, we proceed as follows:

$$\left.\begin{aligned} &\text{Solving (1) for } x, && x=\frac{1+2y}{5}. && (3)\\ &\text{Substituting } \frac{1+2y}{5} \text{ for } x \text{ in (2),} && && \\ & && 4\left(\frac{1+2y}{5}\right)+5y=47. && (4)\end{aligned}\right\}\ (\text{II.})$$

The system (II.) is equivalent to the system (I.), by § 2, Art. 2 (iii.).

$$\left.\begin{aligned} &\text{Solving (4) for } y, && y=7. && (5)\\ &\text{Substituting 7 for } y \text{ in (3),} && x=3. && (6)\end{aligned}\right\}\ (\text{III.})$$

The system (III.) is, by § 2, Art. 2 (iii.), equivalent to the system (II.), and hence to the given system. Therefore the required solution is 3, 7.

In a similar way y could have been first eliminated.

5. The example of the preceding article illustrates the following method of elimination by substitution:

Solve the simpler equation for the unknown number to be eliminated in terms of the other, and substitute the value thus *btained in the other equation. The derived equation will contain ut one unknown number.*

The solution of the given system is then obtained by solving the lerived equation, and substituting the value of the unknown numer thus obtained in the expression for the other unknown number.

EXERCISES III.

Solve the following systems of equations by the method of substitution:

1. $\begin{cases} x = 2y - 3, \\ y = 2x - 15. \end{cases}$ 2. $\begin{cases} x = 3y - 7, \\ y = 3x - 19. \end{cases}$ 3. $\begin{cases} x = \frac{4}{5}y, \\ x - 4 = \frac{3}{5}(y + 6). \end{cases}$

4. $\begin{cases} \frac{1}{2}y - 3x = 2, \\ y = 14x. \end{cases}$ 5. $\begin{cases} x + y = a, \\ x = ny. \end{cases}$ 6. $\begin{cases} 7x = 5y, \\ 15y = 28x - 70. \end{cases}$

7. $\begin{cases} 5x = 8y - 11, \\ 6y = 7x - 21. \end{cases}$ 8. $\begin{cases} 7x - 3 = 5y, \\ 7y - 3 = 8x. \end{cases}$ 9. $\begin{cases} ay = bx, \\ a + y = b + x. \end{cases}$

10. $\begin{cases} x - 3y = 0, \\ 25x + 48y = 287. \end{cases}$ 11. $\begin{cases} 3x + 4y = 2, \\ 9x + 20y = 8. \end{cases}$ 12. $\begin{cases} 5x + 7y = 49, \\ 7x + 5y = 47. \end{cases}$

13. $\begin{cases} \frac{2}{3}x = 10 - \frac{1}{2}y, \\ 4\frac{3}{4}y = 5x - 7. \end{cases}$ 14. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 7, \\ 2x + 3y = 48. \end{cases}$ 15. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 2, \\ 2x + 3y = 60. \end{cases}$

16. $\begin{cases} \dfrac{7 + x}{5} - \dfrac{2x - y}{4} = 3y - 5, \\ \dfrac{5y - 7}{2} + \dfrac{4x - 8}{6} = 18 - 5x. \end{cases}$ 17. $\begin{cases} \left(\dfrac{1}{a} + \dfrac{1}{b}\right)x + \left(\dfrac{1}{a} - \dfrac{1}{b}\right)y = 4, \\ \dfrac{x}{a + b} + \dfrac{y}{a - b} = 2. \end{cases}$

18. $\begin{cases} (x - a)(a + b) = (a - b)(y - a), \\ \dfrac{x}{a^3 - b^3} = \dfrac{y}{a^3 + b^3}. \end{cases}$

Elimination by Comparison.

6. Ex. Solve the system

$$\left.\begin{aligned} 7x + 2y &= 20, \quad (1) \\ 13x - 3y &= 17. \quad (2) \end{aligned}\right\} \qquad \text{(I.)}$$

To eliminate y, we proceed as follows:

$$\left.\begin{aligned} &\text{Solving (1) for } y, \quad y = \frac{20 - 7x}{2}. \quad (3) \\ &\text{Solving (2) for } y, \quad y = \frac{13x - 17}{3}. \quad (4) \end{aligned}\right\} \qquad \text{(II.)}$$

The system (II.) is equivalent to the system (I.), by § 2, Art. 2 (i.). Substituting in (4) for y, its value given in (3),

$$\frac{20-7x}{2}=\frac{13x-17}{3}. \tag{5}$$

The last equation and equation (3) form a system which is equivalent to the system (II.), and hence to the given system.

$$\left.\begin{array}{ll}\text{Solving (5) for } x, & x=2.\\ \text{Substituting 2 for } x \text{ in (3),} & y=3.\end{array}\right\} \quad \text{(III.)}$$

The system (III.) is equivalent to the system formed by equations (3) and (5), and therefore to the given system. Consequently the required solution is 2, 3.

In a similar way, y could have been first eliminated.

7. The example of the preceding article illustrates the following method of elimination by comparison:

Solve the given equations for the unknown number to be eliminated, and equate the expressions thus obtained. The derived equation will contain but one unknown number.

The solution of the given system is then obtained by solving this derived equation, and substituting the value of the unknown number thus obtained in the simplest of the preceding equations.

EXERCISES IV.

Solve the following systems of equations by the method of comparison:

1. $\begin{cases} x=3y-2, \\ x=5y-12. \end{cases}$ **2.** $\begin{cases} y=3x-17, \\ y=2x-10. \end{cases}$ **3.** $\begin{cases} 5y=2x+1, \\ 8y=5x-11. \end{cases}$

4. $\begin{cases} 5x=7y-.1, \\ 7x=9y+1.7. \end{cases}$ **5.** $\begin{cases} \frac{1}{5}x=\frac{1}{3}y-1, \\ \frac{1}{5}y=\frac{1}{4}x-2. \end{cases}$ **6.** $\begin{cases} 2\frac{1}{2}x-3\frac{1}{3}y=10, \\ 7\frac{1}{3}x-5\frac{1}{2}y=55. \end{cases}$

7. $\begin{cases} 1\frac{1}{2}x=7y-38, \\ 1\frac{1}{2}y=7x-72. \end{cases}$ **8.** $\begin{cases} 5x+9y=28, \\ 7x+3y=20. \end{cases}$ **9.** $\begin{cases} 21x-23y=2, \\ 7x-19y=12. \end{cases}$

10. $\begin{cases} \frac{x}{7}+7y=99, \\ \frac{y}{7}+7x=51. \end{cases}$ **11.** $\begin{cases} \frac{x}{2}+\frac{y}{3}-7=0, \\ \frac{x}{3}+\frac{y}{2}-8=0. \end{cases}$ **12.** $\begin{cases} \frac{x}{2}+\frac{y}{6}=11, \\ \frac{x}{5}+\frac{y}{24}=\frac{7}{2}. \end{cases}$

13. $\begin{cases} 8x + 9y = 26, \\ 32x - 3y = 26. \end{cases}$ **14.** $\begin{cases} 63x - 46y = 29, \\ 42x - 69y = 96. \end{cases}$ **15.** $\begin{cases} x + ay + 1 = 0, \\ y + c(x + 1) = 0. \end{cases}$

16. $\begin{cases} 5x + 4y = 9a - b, \\ 7x - 6y = a - 13b. \end{cases}$ **17.** $\begin{cases} ax - by = a^2 + b^2, \\ (a - b)x + (a + b)y = 2(a^2 - b^2). \end{cases}$

The General Solution of a System of Two Linear Equations in Two Unknown Numbers.

8. Any linear equation in two unknown numbers can evidently be brought to the form

$$ax + by = c.$$

9. Let

$$a_1x + b_1y = c_1, \qquad (1)$$

$$a_2x + b_2y = c_2, \qquad (2)$$

be any two linear equations.

Multiplying (1) by b_2, $\quad a_1b_2x + b_1b_2y = b_2c_1. \qquad (3)$

Multiplying (2) by b_1, $\quad a_2b_1x + b_1b_2y = b_1c_2. \qquad (4)$

Subtracting (4) from (3), $\quad (a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2; \qquad (5)$

whence, if $a_1b_2 - a_2b_1 \neq 0$, $\quad x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}. \qquad (6)$

In like manner, if $a_1b_2 - a_2b_1 \neq 0$, $\quad y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \qquad (7)$

But if $a_1b_2 - a_2b_1 = 0$, we have no authority for dividing both members of equation (5) by $a_1b_2 - a_2b_1$.

Consequently, the *general solution* of two linear equations in two unknown numbers is

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

when $a_1b_2 - a_2b_1 \neq 0$.

10. *Every system of two independent and consistent equations of the first degree in two unknown numbers has one, and only one, solution.*

For the system

$$a_1x + b_1y = c_1, \qquad (1)$$

$$a_2x + b_2y = c_2, \qquad (2)$$

is, by Art. 9, equivalent to the system

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2, \qquad (3)$$

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1. \qquad (4)$$

But equations (3) and (4) are each linear in one unknown number, and each, therefore, has one, and only one, solution. Consequently, the given system has one, and only one, solution.

11. ***Three independent linear equations in two unknown numbers cannot be satisfied by any common set of values of the unknown numbers.***

If the values of x and y which constitute the solution of equations (1) and (2), Art. 9, satisfy a third equation,

$$a_3x + b_3y = c_3, \tag{3}$$

we have $$a_3 \times \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} + b_3 \times \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = c_3.$$

From this relation, we obtain

$$\begin{aligned} c_3 &= c_1 \times \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1} + c_2 \times \frac{a_1b_3 - a_3b_1}{a_1b_2 - a_2b_1} \\ &= lc_1 + mc_2, \end{aligned} \tag{i.}$$

wherein $$l = \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1}, \text{ and } m = \frac{a_1b_3 - a_3b_1}{a_1b_2 - a_2b_1}.$$

We also have

$$b_1 \times \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1} + b_2 \times \frac{a_1b_3 - a_3b_1}{a_1b_2 - a_2b_1} = \frac{b_3(a_1b_2 - a_2b_1)}{a_1b_2 - a_2b_1} = b_3.$$

That is, $$b_3 = lb_1 + mb_2. \tag{ii.}$$

In like manner, it can be shown that

$$a_3 = la_1 + ma_2. \tag{iii.}$$

Observe that (ii.) and (iii.) are identities, and hold for all values of the a's and b's which do not reduce $a_1b_2 - a_2b_1$ to 0; while (i.) is not an identity, but imposes a condition upon the values of the known numbers in the three equations.

When this condition is satisfied, c_3 is obtained from c_1 and c_2, just as a_3 is obtained from a_1 and a_2, and b_3 from b_1 and b_2. That is, when the solution of (1) and (2) is also a solution of (3), the last equation is not independent of the other two.

Linear Equations in Three or More Unknown Numbers.

12. The following examples will illustrate the methods for solving systems of three linear equations in three unknown numbers:

Ex. 1. Solve the system

$$2x - 3y + 5z = 11, \tag{1}$$

$$5x + 4y - 6z = -5, \tag{2}$$

$$-4x + 7y - 8z = -14. \tag{3}$$

To eliminate x, we proceed as follows:

Multiplying (1) by 5, $10x - 15y + 25z = 55.$ (4)

Multiplying (2) by 2, $10x + 8y - 12z = -10.$ (5)

Subtracting (4) from (5), $23y - 37z = -65.$ (6)

Multiplying (1) by 2, $4x - 6y + 10z = 22.$ (7)

Adding (3) and (7), $y + 2z = 8.$ (8)

Solving (6) and (8), $y = 2,$

$z = 3.$

Substituting 2 for y and 3 for z in (1), $x = 1.$

Notice that (1), (2), (3); and (1), (6), (8) form equivalent systems. Consequently the required solution is 1, 2, 3.

Ex. 2. Solve the system

$$ay - cz = 0, \tag{1}$$

$$z - x = -b, \tag{2}$$

$$ax + by = a^2 + b(a + c). \tag{3}$$

Notice that by eliminating z from (1) and (2) we obtain an equation in x and y, which with equation (3) gives a system of two equations in the same two unknown numbers.

Solving (2) for z, $z = x - b.$ (4)

Substituting $x - b$ for z in (1),

$$ay - cx + cb = 0. \tag{5}$$

Multiplying (3) by a, $a^2x + aby = a^3 + a^2b + abc.$ (6)

Multiplying (5) by b, $-bcx + aby = -b^2c.$ (7)

Subtracting (7) from (6),
$$\begin{aligned}(a^2 + bc)x &= a^3 + a^2b + abc + b^2c\\ &= a^2(a + b) + bc(a + b)\\ &= (a^2 + bc)(a + b);\end{aligned} \tag{8}$$

whence $x = a + b.$

Substituting $a + b$ for x in (4), $z = a.$

Substituting a for z in (1), $y = c.$

To solve three simultaneous equations in three unknown numbers, eliminate one of the unknown numbers from any two of the equations; next eliminate the same unknown number from the *third equation and either of the other two. Two equations in the same two unknown numbers are thus derived.*

Solve these equations for the two unknown numbers, and substitute the values thus obtained in the simplest equation which contains the third unknown number.

13. From four equations in four unknown numbers, we can by eliminating one of the unknown numbers obtain three equations in three unknown numbers. We then solve these equations for the three unknown numbers and substitute the values thus obtained in the simplest equation which contains the fourth unknown number.

Number of Solutions of a System of Linear Equations.

14. The examples of the preceding articles illustrate the following principles:

(i.) *A system of n independent and consistent linear equations, in n unknown numbers, has one, and only one, determinate solution.*

From the given system a system of $n-1$ equations in $n-1$ unknown numbers can be derived by eliminating one of the unknown numbers. By eliminating from the latter system another unknown number, a second system of $n-2$ equations in $n-2$ unknown numbers is derived; and so on. Finally, a single equation in one unknown number is obtained.

By the principles of equivalent equations the given system is equivalent to a second system which contains the following equations: any one of the given equations in n unknown numbers, any one of the $n-1$ derived equations in $n-1$ unknown numbers, and so on, to any one of the three derived equations in three unknown numbers, either of the two derived equations in two unknown numbers, and the last derived equation in one unknown number.

The last equation in one unknown number has one, and only one, definite solution. If the value of this unknown number be substituted in the next to the last equation of the second system described above, one, and only one, definite value for a second unknown number is obtained. If the values of these two unknown numbers be substituted in the equation in three unknown numbers, one, and only one, definite value of a third unknown number is obtained; and so on.

Consequently the given system is satisfied by one, and only one, definite set of values of the unknown numbers.

(ii.) *A system of n independent linear equations, in more than n unknown numbers, has an indefinite number of solutions.*

For, by each elimination of an unknown number, we derive a set of equations, one less in number, and containing one less unknown number.

Finally, as in (i.), we obtain a single equation. But since the original system contained more unknown numbers than equations, the last derived equation will contain more than one unknown number. Since this equation, therefore, has an indefinite number of solutions, we conclude that the given system has likewise an indefinite number of solutions.

(iii.) *A system of n independent linear equations, in less than n unknown numbers, does not have a determinate finite solution.*

For, if we take from the given system as many equations as there are unknown numbers, the system formed by these equations will have by (i.) one, and only one, definite solution.

But since the other equations of the given system are independent of the equations selected, that is, express independent relations between the unknown numbers, they cannot be satisfied by this solution.

Therefore, the given system cannot be satisfied by any one definite set of values of the unknown numbers.

EXERCISES V.

Solve the following systems of equations:

1. $\begin{cases} x + y = 28, \\ x + z = 30, \\ y + z = 32. \end{cases}$ 2. $\begin{cases} x + y = 2c, \\ x + z = 2b, \\ y + z = 2a. \end{cases}$ 3. $\begin{cases} x - y = 2, \\ y - z = 3, \\ x + z = 9. \end{cases}$

4. $\begin{cases} 3x - y = 7, \\ 3y - z = 5, \\ 3z - x = 0. \end{cases}$ 5. $\begin{cases} 3x + 5y = 35, \\ 3y + 5z = 27, \\ 3z + 5x = 34. \end{cases}$ 6. $\begin{cases} x + y + z = 50, \\ y = 3x - 21, \\ z = 4x - 33. \end{cases}$

7. $\begin{cases} 3x + 2y - 4z = 15, \\ 5x - 3y + 2z = 28, \\ 3y + 4z - x = 24. \end{cases}$ 8. $\begin{cases} x + y - z = 1, \\ 8x + 3y - 6z = 1, \\ 3z - 4x - y = 1. \end{cases}$

9. $\begin{cases} x + y - z = c, \\ x + z - y = b, \\ y + z - x = a. \end{cases}$ 10. $\begin{cases} 4x - 3y + 2z = 9, \\ 2x + 5y - 3z = 4, \\ 5x + 6y - 2z = 18. \end{cases}$

11. $\begin{cases} 2x - 4y + 9z = 28, \\ 7x + 3y - 5z = 3, \\ 9x + 10y - 11z = 4. \end{cases}$ 12. $\begin{cases} x - 2y + 3z = 6, \\ 2x + 3y - 4z = 20, \\ 3x - 2y + 5z = 26. \end{cases}$

13. $$\begin{cases} \dfrac{x}{6}+\dfrac{y}{9}+\dfrac{z}{10}=9, \\[2ex] \dfrac{x}{8}+\dfrac{y}{2}-\dfrac{z}{25}=11, \\[2ex] \dfrac{x}{2}-\dfrac{y}{18}+\dfrac{z}{10}=10. \end{cases}$$

14. $$\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{b+c}=b-a, \\[2ex] \dfrac{y}{c-a}+\dfrac{z}{c+a}=c+a, \\[2ex] \dfrac{x}{b-c}-\dfrac{z}{a-b}=b-c. \end{cases}$$

15. $$\begin{cases} x+y+z=a+b+c, \\ bx+cy+az=a^2+b^2+c^2, \\ cx+ay+bz=a^2+b^2+c^2. \end{cases}$$

16. $$\begin{cases} x+y+z=A, \\ ax+by+cz=0, \\ a^2x+b^2y+c^2z=0. \end{cases}$$

17. $$\begin{cases} (c+a)x-(c-a)y=2bc, \\ (a+b)y-(a-b)z=2ac, \\ (b+c)z-(b-c)x=2ab. \end{cases}$$

18. $$\begin{cases} x+y+z=(a+b+c)^3, \\ ay+bz+cx=3(ab^2+bc^2+ca^2), \\ ax+by+cz=a^3+b^3+c^3+6abc. \end{cases}$$

19. $$\begin{cases} x+y+z=6, \\ x+y+u=7, \\ x+z+u=8, \\ y+z+u=9. \end{cases}$$

20. $$\begin{cases} x+y+z-u=11, \\ x+y-z+u=17, \\ x-y+z+u=9, \\ -x+y+z+u=12. \end{cases}$$

21. $$\begin{cases} 7x-2z+3u=17, \\ 4y-2z+t=15, \\ 5y-3x-2u=8, \\ 4y-3u+2t=17, \\ 3z+8u=33. \end{cases}$$

22. $$\begin{cases} 3x-4y+3z+3v-6u=-1, \\ 3x-5y+2z-4u=-7, \\ 10y-3z+3u-2v=35, \\ 5z+4u+2v-2x=15, \\ 6u-3v+4x-2y=21. \end{cases}$$

§ 4. SYSTEMS OF FRACTIONAL EQUATIONS.

1. If some or all of the equations of a system be fractional, and lead, when cleared of fractions, to linear equations, the solution of the system can be obtained by the preceding methods.

Any solution of the linear system which is derived by clearing of fractions, is a solution of the given system, unless it is a solution of the L. C. D. (equated to 0) of one or more of the fractional equations. (See Ch. X., Art. 4.)

Ex. 1. Solve the system $6x-5y=0,$ (1)

$$\frac{6x+1}{4y+5}=\frac{13}{11}. \qquad (2)$$

Clearing (2) of fractions, $66x+11=52y+65.$ (3)

Transferring and uniting terms in (3), and dividing by 2,

$$33x - 26y = 27. \qquad (4)$$

The solution of (1) and (4) is 15, 18.

In clearing (2) of fractions, we multiplied by $11(4y + 5)$.

Since $x = 15$, $y = 18$ is not a solution of $4y + 5 = 0$, it is a solution of the given system.

Ex. 2. Solve the system

$$\left.\begin{aligned} x + y &= xy, \quad (1)\\ 2x + 2z &= xz, \quad (2)\\ 3y + 3z &= yz. \quad (3)\end{aligned}\right\} \qquad (\text{I.})$$

Observe that the given equations are neither linear nor fractional. Yet they can be transformed so that they will contain only the reciprocals of x, y, and z.

Dividing (1) by xy, (2) by xz, (3) by yz, we have:

$$\frac{1}{y} + \frac{1}{x} = 1. \quad (4) \qquad \frac{2}{z} + \frac{2}{x} = 1. \quad (5) \qquad \frac{3}{z} + \frac{3}{y} = 1. \quad (6) \qquad (\text{II.})$$

We will solve this system for $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

Multiplying (4) by 2, $\qquad \frac{2}{y} + \frac{2}{x} = 2. \qquad (7)$

Subtracting (5) from (7), $\qquad \frac{2}{y} - \frac{2}{z} = 1. \qquad (8)$

Solving (6) and (8) for $\frac{1}{y}$ and $\frac{1}{z}$, $\quad \frac{1}{y} = \frac{5}{12}$, $\frac{1}{z} = -\frac{1}{12}$.

Substituting $\frac{5}{12}$ for $\frac{1}{y}$ in (4), $\qquad \frac{1}{x} = \frac{7}{12}$.

Consequently, a solution of the given system is $\frac{12}{7}, \frac{12}{5}, -12$.

It is important to notice that we cannot assume that the system (II.) is equivalent to the system (I.), since the equations of (II.) are derived from the equations of (I.) by dividing by expressions which contain the unknown numbers.

But if any solution of (I.) be lost by this transformation, it must be a solution of the expressions (equated to 0) by which the equations of (I.) were divided; that is, of

$$xy = 0, \; xz = 0, \; yz = 0. \qquad (\text{III.})$$

The system (III.) has the solution 0, 0, 0, and this solution evidently satisfies the system (I.).

We therefore conclude that the given system has the two solutions $\frac{12}{7}$, $\frac{12}{5}$, -12, and 0, 0, 0.

Ex. **3.** Solve the system

$$\frac{3}{x+y+z}+\frac{6}{2x-y}+\frac{1}{y-3z}=1, \tag{1}$$

$$\frac{6}{x+y+z}+\frac{4}{2x-y}-\frac{1}{y-3z}=3, \tag{2}$$

$$\frac{15}{x+y+z}-\frac{2}{2x-y}-\frac{3}{y-3z}=5. \tag{3}$$

This system can be readily solved by making the following substitutions:

Let $$\frac{3}{x+y+z}=u,\ \frac{2}{2x-y}=v,\ \frac{1}{y-3z}=w. \tag{I.}$$

Then the given system becomes

$$u+3v+w=1, \tag{4}$$

$$2u+2v-w=3, \tag{5}$$

$$5u-v-3w=5. \tag{6}$$

Solving equations (4), (5), and (6), we obtain

$$u=\tfrac{1}{2},\ v=\tfrac{1}{2},\ w=-1.$$

Substituting these values in the system (I.), we have

$$x+y+z=6,\ 2x-y=4,\ y-3z=-1.$$

Whence, $x=3,\ y=2,\ z=1.$

2. As in a system of integral equations, so in a system of fractional equations, the equations must be consistent and independent.

Ex. Solve the system $$3x+4y=11, \tag{1}$$

$$\frac{1}{x-1}+\frac{1}{y-2}=0. \tag{2}$$

Clearing (2) of fractions and uniting terms,

$$x+y=3. \tag{3}$$

Solving (1) and (3), $x=1,\ y=2.$

These values constitute a solution of equations (1) and (3), but not of (1) and (2). For they form a solution of the L. C. D. (equated to 0) of the fractions in (2); that is, of

$$(x-1)(y-2)=0. \tag{4}$$

We conclude, therefore, that equations (1) and (2) do not have any solution.

It can, in fact, be shown that they are inconsistent. For (1) is equivalent to

$$3x-3+4y-8=0,$$

or to
$$3(x-1)+4(y-2)=0.$$

Dividing both members of the last equation by $(x-1)(y-2)$, we obtain

$$\frac{3}{y-2}+\frac{4}{x-1}=0. \tag{5}$$

Equation (5) is evidently inconsistent with (2).

It should be noticed that in clearing (2) of fractions no unnecessary factor was used. The explanation of the apparent contradiction of the principle proved in Ch. X., Art. 3, is that this principle holds only when the fractional equation contains but one unknown number.

EXERCISES VI.

Solve the following systems of equations:

1. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{6}. \end{cases}$ **2.** $\begin{cases} \dfrac{3}{x}+\dfrac{5}{y}=2, \\ \dfrac{9}{x}-\dfrac{10}{y}=1. \end{cases}$ **3.** $\begin{cases} 7x-\dfrac{3}{y}=16, \\ 3x-\dfrac{2}{y}=4. \end{cases}$

4. $\begin{cases} \dfrac{4}{x}-3y=8, \\ \dfrac{5}{x}-6y=1. \end{cases}$ **5.** $\begin{cases} \dfrac{14}{x}+\dfrac{25}{y}=7, \\ \dfrac{49}{x}-\dfrac{50}{y}=-3. \end{cases}$ **6.** $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x}-\dfrac{1}{y}=b. \end{cases}$

7. $\begin{cases} \dfrac{p}{x}+\dfrac{q}{y}=a, \\ \dfrac{q}{x}+\dfrac{p}{y}=b. \end{cases}$ **8.** $\begin{cases} 6y-6x=xy, \\ 10y+3x=6xy. \end{cases}$ **9.** $\begin{cases} \dfrac{3}{x-4}+\dfrac{4}{y-1}=3, \\ \dfrac{9}{x-4}-\dfrac{2}{y-1}=2. \end{cases}$

10. $\begin{cases} \dfrac{3x+7}{10y+3}=1, \\ \dfrac{12x+5}{7y+1}=2. \end{cases}$ **11.** $\begin{cases} \dfrac{3}{2x-3y}+\dfrac{5}{y-2}=8, \\ \dfrac{7}{2x-3y}+\dfrac{3}{y-2}=10. \end{cases}$

12. $$\begin{cases} \dfrac{x+y-1}{x-y+1}=a, \\ \dfrac{y-x+1}{x-y+1}=ab. \end{cases}$$

13. $$\begin{cases} \dfrac{7}{2x+3y}=\dfrac{11}{2x-3y}, \\ \dfrac{x}{10y-7}=\dfrac{9}{10}. \end{cases}$$

14. $$\begin{cases} \dfrac{x+a-b}{y-a-b}=\dfrac{x-b}{y-a}, \\ \dfrac{b}{x-a}=\dfrac{a}{y+b}. \end{cases}$$

15. $$\begin{cases} \dfrac{2n}{x+ny}-\dfrac{1}{n-ny}=1, \\ \dfrac{10n}{x+ny}+\dfrac{3}{n-ny}=1. \end{cases}$$

16. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, \\ \dfrac{1}{x}+\dfrac{1}{z}=6, \\ \dfrac{1}{y}+\dfrac{1}{z}=7. \end{cases}$$

17. $$\begin{cases} \dfrac{1}{z}+\dfrac{1}{y}=a, \\ \dfrac{1}{z}+\dfrac{1}{x}=b, \\ \dfrac{1}{x}+\dfrac{1}{y}=c. \end{cases}$$

18. $$\begin{cases} \dfrac{1}{x}+\dfrac{2}{y}+\dfrac{2}{z}=16, \\ \dfrac{1}{y}+\dfrac{2}{z}+\dfrac{2}{x}=15, \\ \dfrac{1}{z}+\dfrac{2}{x}+\dfrac{2}{y}=14. \end{cases}$$

EXERCISES VII.

Solve the following systems of equations by the methods given in this chapter:

1. $$\begin{cases} 22x-46y=126, \\ 25x+69y=391. \end{cases}$$

2. $$\begin{cases} a(x+y)-b(x-y)=2a^2, \\ (a^2-b^2)(x-y)=4a^2b. \end{cases}$$

3. $$\begin{cases} \dfrac{x-7}{3}+\dfrac{y-5}{2}=7, \\ \dfrac{x-7}{2}+\dfrac{y-5}{3}=8. \end{cases}$$

4. $$\begin{cases} \dfrac{2x+7y}{4}-\dfrac{x+7}{6}=4, \\ \dfrac{2x+7y}{6}-\dfrac{x+7}{3}=0. \end{cases}$$

5. $$\begin{cases} \dfrac{2x+1}{5}-\dfrac{3y+2}{7}=2y-x, \\ \dfrac{3x-1}{4}+\dfrac{7y+2}{6}=2x-y. \end{cases}$$

6. $$\begin{cases} \dfrac{3x-4}{2}+\dfrac{4y-1}{5}=x+y, \\ \dfrac{5x-9}{7}-\dfrac{y-2}{2}=x-y. \end{cases}$$

7. $$\begin{cases} \dfrac{x}{n+1}+\dfrac{y}{n-1}=\dfrac{1}{n-1}, \\ \dfrac{x}{n-1}+\dfrac{y}{n+1}=\dfrac{1}{n^2-1}. \end{cases}$$

8. $$\begin{cases} \dfrac{x-1}{y-1}=\dfrac{3}{4}, \\ \dfrac{x+3}{y+3}=\dfrac{10}{13}. \end{cases}$$

9. $$\begin{cases} \dfrac{ax+by}{2}+x=\dfrac{a+1}{a}, \\ \dfrac{ax+by}{2}+y=\dfrac{b+1}{b}. \end{cases}$$

10. $$\begin{cases} \dfrac{1}{2}(x+y)=1+\dfrac{x-y}{2a}, \\ \dfrac{a}{2}(x-y)=1+\dfrac{x-y}{2a}. \end{cases}$$

11. $$\begin{cases} a^2x-b^2y=0, \\ (a^2+b^2)x+(a^2-b^2)y=a^4+b^4. \end{cases}$$

12. $$\begin{cases} (a+b)x+(a-b)y=a^2+b^2, \\ (a-b)x+(a+b)y=a^2-b^2. \end{cases}$$

13. $\begin{cases} x + y = z + 10, \\ y = 2x - 13, \\ z = 2y - 11. \end{cases}$

14. $\begin{cases} yz = 2(y + z), \\ xz = 3(x + z), \\ xy = 4(x + y). \end{cases}$

15. $$\begin{cases} x - \dfrac{3x + 5y}{17} + 17 = 5y + \dfrac{4x + 7}{3}, \\ \dfrac{22 - 6y}{3} - \dfrac{5x - 7}{11} = \dfrac{x + 1}{6} - \dfrac{8y + 5}{18}. \end{cases}$$

16. $$\begin{cases} \dfrac{3x + 7y + 1}{5} - \dfrac{2x - 3y + 8}{3} = 2, \\ \dfrac{5x - 7y + 10}{3} - \dfrac{3x + 2y + 6}{5} = 2. \end{cases}$$

17. $$\begin{cases} \dfrac{10}{2x + 3y - 29} + \dfrac{9}{7x - 8y + 24} = 8, \\ \dfrac{2x + 3y - 29}{2} = \dfrac{7x - 8y}{3} + 8. \end{cases}$$

18. $$\begin{cases} \frac{1}{2}(a + b - c)x + \frac{1}{2}(a - b + c)y = a^2 + (b - c)^2, \\ \frac{1}{2}(a - b + c)x + \frac{1}{2}(a + b - c)y = a^2 - (b - c)^2. \end{cases}$$

19. $$\begin{cases} \dfrac{x}{n^2-1} - \dfrac{y}{a^2-1} = a^2 - n^2, \\ \dfrac{x}{a^2+1} + \dfrac{y}{n^2+1} = a^2 + n^2 - 2. \end{cases}$$

20. $$\begin{cases} \dfrac{y - 6}{x - 4} - \dfrac{10}{16 - x^2} = \dfrac{y + 6}{x + 4}, \\ \dfrac{5}{x^2 - 3x} + \dfrac{3}{3y - xy} = -\dfrac{10}{xy}. \end{cases}$$

21. $$\begin{cases} \dfrac{xy}{x + y} = a, \\ \dfrac{xz}{x + z} = b, \\ \dfrac{yz}{y + z} = c. \end{cases}$$

22. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = a, \\ \dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z} = b, \\ \dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -c. \end{cases}$$

23. $$\begin{cases} (x - 1)(4y + 3) = (4x - 8)(y + 2), \\ (x - 2)(3z + 1) = (3x - 8)(z + 1), \\ (y + 1)(2z + 3) = (2y + 1)(z + 2). \end{cases}$$

24. $xyz = a(yz - zx - xy) = b(zx - xy - yz) = c(xy - yz - zx).$

25. $\begin{cases} x + ay + a^2z + a^3 = 0, \\ x + by + b^2z + b^3 = 0, \\ x + cy + c^2z + c^3 = 0. \end{cases}$

26. $\begin{cases} ax + by + cz = A, \\ a^2x + b^2y + c^2z = A^2, \\ a^3x + b^3y + c^3z = A^3. \end{cases}$

27. $\begin{cases} x + y + z = a + b + c, \\ bx + cy + az = cx + ay + bz = 0. \end{cases}$

28.
$$\begin{cases} \dfrac{bx+ay}{c}=\dfrac{a-b}{(a-c)(b-c)}, \\ \dfrac{cy+bz}{a}=\dfrac{b-c}{(b-a)(c-a)}, \\ \dfrac{az+cx}{b}=\dfrac{c-a}{(a-b)(c-b)}. \end{cases}$$

29.
$$\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{b-c}+\dfrac{z}{c+a}=2c, \\ \dfrac{x}{a-b}-\dfrac{y}{b-c}+\dfrac{z}{c-a}=2a, \\ \dfrac{x}{a-b}-\dfrac{y}{b-c}-\dfrac{z}{c+a}=2a-2c. \end{cases}$$

30.
$$\begin{cases} (4-x)(244-y)=z, \\ (7-x)(124-y)=z, \\ (13-x)(64-y)=z. \end{cases}$$

31.
$$\begin{cases} (z+x)a-(z-x)b=2yz, \\ (x+y)b-(x-y)c=2xz. \\ (y+z)c-(y-z)a=2xy. \end{cases}$$

32.
$$\begin{cases} x+y+z=0, \\ (b+c)x+(c+a)y+(a+b)z=0, \\ bcx+cay+abz=1. \end{cases}$$

33.
$$\begin{cases} \dfrac{5x+7y+2}{3}-\dfrac{3x+4y+7}{4}=x, \\ \dfrac{7x+3y+4}{4}-\dfrac{6x+5y+7}{5}=y. \end{cases}$$

34.
$$\begin{cases} \dfrac{4x-3y+19}{4}-\dfrac{x-2y+9}{6}=x, \\ \dfrac{5x-4y+21}{6}-\dfrac{3x-2y-2}{9}=y. \end{cases}$$

35.
$$\begin{cases} \dfrac{8}{2x-3y+17}+5x-8y+39=0, \\ \dfrac{5}{2x-3y+17}+16y-10x=88\frac{1}{2}. \end{cases}$$

36.
$$\begin{cases} (a-b)x+(b-c)y+(c-a)z=2(a^2+b^2+c^2-ab-ac-bc) \\ (a-b)y+(b-c)z+(c-a)x=ab+ac+bc-a^2-b^2-c^2, \\ x+y+z=0. \end{cases}$$

37.
$$\begin{cases} yz+xz+xy=xyz, \\ yu+xu+xy=xyu, \\ zu+xu+xz=xuz, \\ zu+yu+yz=uyz. \end{cases}$$

38.
$$\begin{cases} a^4+a^3x+a^2y+az+u=0. \\ b^4+b^3x+b^2y+bz+u=0, \\ c^4+c^3x+c^2y+cz+u=0. \\ d^4+d^3x+d^2y+dz+u=0. \end{cases}$$

39.
$$\begin{cases} x+y+z+u=16, \\ x+y+z+v=18, \\ x+y+u+v=20, \\ x+z+u+v=22, \\ y+z+u+v=24. \end{cases}$$

40.
$$\begin{cases} 2u-3v=2a-7b+2c, \\ \quad v+2z=7b, \\ 3z+\quad y=3a+6b, \\ 4y-2x=8a, \\ 3x-5u=a-5b-5c. \end{cases}$$

CHAPTER XIV.

PROBLEMS.

1. As was stated in Ch. V., Art. 2 (ii.), every problem which can be solved must contain as many conditions, expressed or implied, as there are required numbers. In solving a problem by means of one equation in one unknown number, one of the required numbers was usually taken as the unknown number of the equation. All but one of the conditions of the problem were used to express the other required numbers in terms of the one selected as the unknown number. The remaining condition furnished the equation of the problem.

But a problem which contains more than one condition can be solved by means of a system of equations in which the unknown numbers are usually the required numbers of the problem. Each condition then furnishes an equation. The solution of the system of equations thus obtained gives the solution of the problem, if the conditions of the latter be consistent.

2. We will first solve by means of a system of two equations one of the problems which was solved in Ch. V. by means of one equation in one unknown number.

Pr. **1.** (Pr. 4, Ch. V.) At an election at which 943 votes were cast, A and B were candidates. A received a majority of 65 votes. How many votes were cast for each candidate?

Let x stand for the number of votes cast for A,

and y for the number of votes cast for B.

Then, by the first condition,

$$x + y = 943, \qquad (1)$$

and by the second condition,

$$x - y = 65, \qquad (2)$$

Whence $x = 504$, the number of votes cast for A,

$y = 439$, the number of votes cast for B.

Had we substituted the value of y obtained from (1), namely $943 - x$, for y in (2), we should have obtained the equation of the solution in Ch. V.,

$$x - (943 - x) = 65.$$

Pr. 2. A tank can be filled by two pipes. If the first be left open 6 minutes, and the second 7 minutes, the tank will be filled; or if the first be left open 3 minutes, and the second 12 minutes, the tank will be filled. In what time can each pipe fill the tank?

Let x stand for the number of minutes it takes the first pipe to fill the tank, and y for the number of minutes it takes the second pipe. Let the capacity of the tank be represented by 1.

Then in 1 minute the first pipe fills $\frac{1}{x}$ of the tank, and in 6 minutes $\frac{6}{x}$ of the tank; the second pipe fills $\frac{7}{y}$ of the tank in 7 minutes. Therefore, by the conditions of the problem,

$$\frac{6}{x} + \frac{7}{y} = 1; \quad \frac{3}{x} + \frac{12}{y} = 1.$$

Whence $x = 10\frac{1}{5}$, $y = 17$.

EXERCISES I.

1. Find two numbers whose sum is 19 and whose difference is 7.

2. If one number be multiplied by 3 and another by 7, the sum of the products will be 58; if the first be multiplied by 7 and the second by 3, the sum will be 42. What are the numbers?

3. In a meeting of 48 persons, a motion was carried by a majority of 18. How many persons voted for the motion and how many against it?

4. If one of two numbers be divided by 6 and the other by 5, the sum of the quotients will be 52; if the first be divided by 8 and the second by 12, the sum of the quotients will be 31. What are the numbers?

5. Find two numbers, such that if 1 be subtracted from the first and added to the second the results will be equal; while if 5 be subtracted from the first and the second be subtracted from 5, these results will also be equal.

6. If 45 be subtracted from a number, the remainder will be a certain multiple of 5; but if the number be subtracted from 135, the remainder will be the same multiple of 10. What is the number, and what multiple of 5 is the first remainder?

7. If 1 be added to the numerator of a fraction, the resulting fraction will be equal to $\frac{1}{4}$; but if 1 be added to the denominator, the resulting fraction will be equal to $\frac{1}{5}$. What is the fraction?

8. If 1 be subtracted from the numerator and denominator of a certain fraction, the resulting fraction will be equal to $\frac{1}{3}$; but if 1 be added to the numerator and denominator of the same fraction, the resulting fraction will be equal to $\frac{1}{2}$. What is the fraction?

9. A said to B: "Give me three-fourths of your marbles and I shall have 100 marbles." B said to A: "Give me one-half of your marbles and I shall have 100 marbles." How many marbles had A and B?

10. A bag contains white and black balls. One-half of the number of white balls is equal to one-third of the number of black balls, and twice the number of white balls is 6 less than the total number of balls. How many balls of each color are there?

11. The sum of two numbers is 47. If the greater be divided by the less, the quotient and the remainder will each be 5. What are the numbers?

12. A father said to his son: "After 3 years I shall be three times as old as you will be, and 7 years ago I was seven times as old as you then were." What were the ages of father and son?

13. A merchant received from one customer $26 for 10 yards of silk and 4 yards of cloth; and from another customer $23 for 7 yards of silk and 6 yards of cloth at the same prices. What was the price of the silk and of the cloth?

14. A merchant has two kinds of wine. If he mix 9 gallons of the poorer with 7 gallons of the better, the mixture will be worth $1.37½ a gallon; but if he mix 3 gallons of the poorer with 5 gallons of the better, the mixture will be worth $1.45 a gallon. What is the price of each kind of wine?

15. A man has a gold watch, a silver watch, and a chain. The gold watch and the chain cost seven times as much as the silver watch; the cost of the chain and half the cost of the silver watch is equal to three-tenths of the cost of the gold watch. If the chain cost $40, what was the cost of each watch?

16. A and B make a purchase for $48. A gives all of his money, and B three-fourths of his. If A had given three-fourths of his money and B all of his, they would have paid $1.50 less. How much money had A and B?

17. A mechanic and an apprentice together receive $40. The mechanic works 7 days and the apprentice 12 days; and the mechanic earns in 3 days $7 more than the apprentice earns in 5 days. What wages does each receive?

18. I have 7 silver balls equal in weight and 12 gold balls equal in weight. If I place 3 silver balls in one pan of a balance and 5 gold balls in the other, I must add to the gold balls 7 ounces to maintain equilibrium. If I place in one pan 4 silver balls and in the other 7 gold balls, the balance is in equilibrium. What is the weight of each gold and of each silver ball?

19. A tank has two pumps. If the first be worked 2 hours and the second 3 hours, 1020 cubic feet of water will be discharged. But if the first be worked 1 hour and the second $2\frac{1}{2}$ hours, 690 cubic feet of water will be discharged. How many cubic feet of water can each pump discharge in one hour?

20. It was intended to distribute $25 among a certain number of the poor, each adult to receive $2.50 and each child 75 cents. But it was found that there were 3 more adults and 5 more children than was at first supposed. Each adult was therefore given $1.75 and each child 50 cents. How many adults and how many children were there?

21. A man ordered a wine-merchant to fill two casks of different sizes with wine, one at $1.20 and the other at $1.50 a quart, paying $88.50 for both casks of wine. By mistake the casks were interchanged, so that the purchaser received more of the cheaper wine and less of the dearer. The merchant therefore returned to him $1.50. How many quarts did each cask hold?

22. A and B jointly contribute $10,000 to a business. A leaves his money in the business 1 year and 3 months, and B his money 2 years and 11 months. If their profits be equal, how much does each contribute?

23. A merchant sold 12 gallons from each of two full casks of wine, and then found that the larger contained twice as much as the smaller. After he had sold more wine from both casks, he found that each one contained one-third of its original capacity. If he had then added 4 gallons of wine to each cask, the contents of the smaller would have been three-fourths of the contents of the larger. What was the capacity of each cask?

24. One boy said to another: "Give me 5 of your nuts, and I shall have three times as many as you will have left." "No," said the other, "give me 2 of your nuts, and I shall have five times as many as you will have left." How many nuts had each boy?

25. A father has two sons, one 4 years older than the other. After 2 years the father's age will be twice the joint ages of his sons; and 6 years ago his age was six times the joint ages of his sons. How old is the father and each of his sons?

26. If a number of two digits be divided by the sum of the digits, the quotient will be 7. If the digits be interchanged, the resulting number will be less than the original number by 27. What is the number?

27. A man walks 26 miles, first at the rate of 3 miles an hour, and later at the rate of 4 miles an hour. If he had walked 4 miles an hour when he walked 3, and 3 miles an hour when he walked 4, he would have gone 4 miles further. How far would he have gone, if he had walked 4 miles an hour the whole time?

28. Two trains leave different cities, which are 650 miles apart, and run toward each other. If they start at the same time, they will meet after 10 hours; but if the first start $4\frac{1}{3}$ hours earlier than the second, they will meet 8 hours after the second train starts. What is the speed of each train?

29. If the base of a rectangle be increased by 2 feet, and the altitude be diminished by 3 feet, the area will be diminished by 48 square feet. But if the base be increased by 3 feet, and the altitude be diminished by 2 feet, the area will be increased by 6 square feet. Find the base and the altitude of the rectangle.

30. A number of three digits is in value between 400 and 500, and the sum of its digits is 9. If the digits be reversed, the resulting number will be $\frac{36}{47}$ of the original number. What is the number?

31. The report of a cannon travels with the wind 344.42 yards a second, and against the wind 335.94 yards a second. What is the velocity of the report in still air, and what is the velocity of the wind?

32. Two messengers, A and B, travel toward each other, starting from two cities which are 805 miles distant from each other. If A start $5\frac{3}{4}$ hours earlier than B, they will meet $6\frac{1}{8}$ hours after B starts. But if B start $5\frac{3}{4}$ hours earlier than A, they will meet $5\frac{5}{8}$ hours after A starts. At what rates do A and B travel?

33. Each of two servants was to receive $160, a dress, and a pair of shoes for one year's services. One servant left after 8 months, and received the dress and $106; the other servant left after $9\frac{1}{2}$ months, and received a pair of shoes and $142. What was the value of the dress, and of the pair of shoes?

34. On the eve of a battle, one army had 5 men to every 6 men in the other. The first army lost 14,000 men, and the second lost 6000 men. The first army then had 2 men to every 3 men in the other. How many men were there originally in each army?

35. If the sum of two numbers, each of three digits, be increased by 1, the result will be 1000. If the greater be placed on the left of the less, and a decimal point be placed between them, the resulting number will be six times the number obtained by placing the smaller number on the left of the greater, with a decimal point between them. What are the numbers?

36. A vessel sails 110 miles with the current and 70 miles against the current in 10 hours. On a second trip, it sails 88 miles with the current and 84 miles against the current in the same time. How many miles can the vessel sail in still water in one hour, and what is the speed of the current?

37. A and B run a race of 400 yards. In the first heat A gives B a start of 20 seconds, and wins by 50 yards. In the second heat A gives B a start of 125 yards, and wins by 5 seconds. What is the speed of each runner?

38. A merchant had two casks containing different quantities of wine. He poured from the first cask into the second as much wine as was in the second; next he poured from the second cask into the first as much wine as was left in the first; finally he poured from the first cask into the second as much wine as was left in the second. Each cask then contained 80 quarts. How many gallons did each cask originally contain?

39. A and B formed a partnership. A invested $20,000 of his own money and $5000 which he borrowed; B invested $22,000 of his own money and $8000 which he borrowed at the same rate of interest as was paid by A. At the end of a year, A's share in the profits amounted to $1750 more than the interest on his $5000, and B's share to $2000 more than the interest on his $8000. What rate per cent interest did they pay, and what rate per cent did they realize on their investments?

40. Two bodies move along the circumference of a circle in the same direction from two different points, the shorter distance between which, measured along the circumference, is 160 feet. One body will overtake the other in 32 seconds, if they move in one direction; or in 40 seconds, if they move in the opposite direction. While the one goes once around the circumference, the distance passed over by the other exceeds the circumference by 45 feet. What is the circumference of the circle, and at what rates do the bodies move?

41. A number of workmen, who receive the same wages, earn together a certain sum. Had there been 7 more workmen, and had each one received 25 cents more, their joint earnings would have increased by $18.65. Had there been 4 fewer workmen, and had each one received 15 cents less, their joint earnings would have decreased by $9.20. How many workmen are there, and how much does each one receive?

42. A courier rode from *A* toward *B*, which is 64 miles distant from *A*. Five hours after his departure, a second courier started from *B* and rode toward *A*. The couriers met 7 hours after the second courier started. If the second courier had started from *B* 2 hours before the first started from *A*, they would have met 8 hours after the second courier started. At what rate did each courier ride?

43. A farmer has enough feed for his oxen to last a certain number of days. If he were to sell 75 oxen, his feed would last 20 days longer. If, however, he were to buy 100 oxen, his feed would last 15 days less. How many oxen has he, and for how many days has he enough feed?

44. An alloy of tin and lead, weighing 40 pounds, loses 4 pounds in weight when immersed in water. Find the amount of tin and lead in the alloy, if 10 pounds of tin lose $1\frac{3}{8}$ pounds when immersed in water, and 5 pounds of lead lose .375 of a pound.

45. Two men were to receive $96 for a certain piece of work, which they could do together in 30 days. After half of the work was done, one of them stopped for 8 days, and then the other stopped for 4 days. They finally completed the work in $35\frac{1}{2}$ days. How many dollars should each one receive, and in what time could each one have done the work alone?

46. Two boys, A and B, run a race from *P* to *Q* and return. A, the faster runner, on his return meets B 90 feet from *Q*, and reaches *P* 3 minutes ahead of B. If he had run again to *Q*, he would have met B at a distance from *P* equal to one-sixth of the distance from *P* to *Q*. How far is *Q* from *P*, and how long did it take B to run from *P* to *Q* and return?

47. It took a certain number of workmen 6 hours to carry a pile of stones from one place to another. Had there been 2 more workmen, and had each one carried 4 pounds more at each trip, it would have taken them 1 hour less to complete the work. Had there been 3 fewer workmen, and had each one carried 5 pounds less at each trip, it would have taken them 2 hours longer to complete the work. How many workmen were there, and how many pounds did each one carry at every trip?

48. Three carriages travel from *A* to *B*. The second carriage travels every 4 hours 1 mile less than the first, and is 4 hours longer in making the journey. The third carriage travels every 3 hours $1\frac{3}{4}$ miles more than the second, and is 7 hours less in making the journey. How far is *B* from *A*, and how many hours does it take each carriage to make the journey?

49. Water enters a basin through one pipe and is discharged through another. Through the first pipe four more gallons enter the basin every minute than is discharged through the second. When the basin is empty, both pipes are opened, the first one hour earlier than the second, and after a certain time the basin contains 1760 gallons. The pipe through which water enters is then closed, and after one hour is again opened. If both pipes be then left open as long as they were open together in the former case, the basin will contain 880 gallons. In what time can the one pipe fill the basin and the other empty it, if it hold 1760 gallons?

50. A body moves with a uniform velocity from a point A to a point B, which is 323 feet distant from A, and without stopping returns at the same rate from B to A. A second body leaves B 13 seconds after the first leaves A, and moves toward A with a uniform but less velocity than the velocity of the first. The first body meets the second 10 seconds after the latter starts, and on returning to A overtakes the second body 45 seconds after the latter starts. What is the velocity of each body?

51. A fox pursued by a dog is 60 of her own leaps ahead of the dog. The fox makes 9 leaps while the dog makes 6, but the dog goes as far in 3 leaps as the fox goes in 7. How many leaps does each make before the dog catches the fox?

52. The sum of the three digits of a number is 14; the sum of the first and the third digit is equal to the second; and if the digits in the units' and in the tens' place be interchanged, the resulting number will be less than the original number by 18. What is the number?

53. The sum of the ages of A, B, and C is 69 years. Two years ago B's age was equal to one-half of the sum of the ages of A and C, and 10 years hence the sum of the ages of B and C will exceed A's age by 31 years. What are the present ages of A, B, and C?

54. The total capacity of three casks is 1440 quarts. Two of them are full and one is empty. To fill the empty cask it takes all the contents of the first and one-fifth of the contents of the second, or the contents of the second and one-third of the contents of the first. What is the capacity of each cask?

55. Three brothers wished to buy a house worth $70,000, but none of them had enough money. If the oldest brother had given the second brother one-third of his money, or the youngest brother one-fourth of his money, each of the latter would then have had enough money to buy the house. But the oldest brother borrowed one-half of the money of the youngest and bought the house. How much money had each brother?

56. The sum of the three digits of a number is 9. The digit in the hundreds' place is equal to one-eighth of the number composed of the two other digits, and the digit in the units' place is equal to one-eighth of the number composed of the two other digits. What is the number?

57. Find the contents of three vessels from the following data: If the first be filled with water and the second be filled from it, the first will then contain two-thirds of its original contents; if from the first, when full, the third be filled, the first will then contain five-ninths of its original contents; finally, if from the first, when full, the second and third be filled, the first will then contain 8 gallons.

58. Three boys were playing marbles. A said to B: "Give me 5 marbles, and I shall have twice as many as you will have left." B said to C: "Give me 13 marbles, and I shall have three times as many as you will have left." C said to A: "Give me 3 marbles, and I shall have six times as many as you will have left." How many marbles did each boy have?

59. Three cities, *A*, *B*, and *C*, are situated at the vertices of a triangle. The distance from *A* to *C* by way of *B* is 82 miles, from *B* to *A* by way of *C* is 97 miles, and from *C* to *B* by way of *A* is 89 miles. How far are *A*, *B*, and *C* from one another?

60. A father's age is twenty-one times the difference between the ages of his two sons. Six years ago his age was six times the sum of his sons' ages, and two years hence it will be twice the sum of their ages. Find the ages of the father and his two sons.

61. A regiment of 600 soldiers is quartered in a four-story building. On the first floor are twice as many men as are on the fourth; on the second and third are as many men as are on the first and fourth; and to every 7 men on the second there are 5 on the third. How many men are quartered on each floor?

62. The sum of the three digits of a number is 9. If 198 be added to the number, the digits of the resulting number are those of the given number written in reverse order. Two-thirds of the digit in the tens' place is equal to the difference between the digits in the units' and in the hundreds' place. What is the number?

63. Four men are to do a piece of work. A and B can do the work in 10 days, A and C in 12 days, A and D in 20 days, and B, C, and D in $7\frac{1}{2}$ days. In how many days can each man do the work, and in how many days can they all together do the work?

64. The year in which printing was invented is expressed by a number of four digits, whose sum is 14. The tens' digit is one-half of the units' digit, and the hundreds' digit is equal to the sum of the thousands' and the tens' digit. If the digits be reversed, the resulting number will be equal to the original number increased by 4905. In what year was printing invented?

Discussion of Solutions.

3. Pr. 1. A merchant has two kinds of tea; the first is worth a cents a pound, and the second b cents a pound. How much of each kind must be taken to make a mixture of one pound worth c cents?

Let x stand for the part of a pound of the first kind, and y for the part of a pound of the second kind.

Then, by the first condition, $x + y = 1;$ (1)

and by the second condition, $ax + by = c.$ (2)

Whence $x = \frac{c - b}{a - b}, \quad y = \frac{a - c}{a - b}.$

(i.) If $a > c > b$, the values of x and y are both positive, and the solution satisfies the conditions of the problem. Thus, if $a = 100$, $b = 75$, and $c = 85$, we have $x = \frac{2}{5}$, $y = \frac{3}{5}$.

If $a < c < b$, then x and y are both positive, and satisfy the conditions of the problem. That is, if the value of the pound of mixture be intermediate between the values of a pound of each of the two kinds, a definite solution is always possible.

(ii.) If $c > a > b$, then x will be ***positive*** and y ***negative***. The solution does not satisfy the conditions of the problem. Thus, if $a = 100$, $b = 75$, $c = 110$, we obtain $x = \frac{7}{5}$, $y = -\frac{2}{5}$.

It is evident that a one-pound mixture of two kinds of tea which is worth more than either kind cannot be made.

(iii.) If $a = b = c$, then $x = \frac{0}{0}$, $y = \frac{0}{0}$. This solution shows that the conditions of the problem may be satisfied in an indefinite number of ways. It is evident that a one-pound mixture of two kinds of tea, that are the same in price, can be made in any number of ways, if the mixture be the same in price.

(iv.) If $a = b$, and $a \neq c$, then $x = \infty$ and $y = \infty$.

This solution does not satisfy the conditions of the problem, since x and y must be finite proper fractions. It is also evident that a one-pound mixture of two kinds of tea which are the same in price cannot be made, if the mixture is to be of a different price.

EXERCISES II.

Solve the following problems, and discuss the results:

1. If an alloy of two kinds of silver be made, and a ounces of the first be taken with b ounces of the second, the mixture will be worth m dollars an ounce. If b ounces of the first be taken with a ounces of the second, the mixture will be worth n dollars an ounce. How much is an ounce of each kind of silver worth?

2. Two bodies are separated by a distance of d yards. If they move toward each other with different velocities, they will meet after m seconds; but if they both move in the same direction, the one will overtake the other after n seconds. With what velocities do the bodies move?

CHAPTER XV.

INDETERMINATE LINEAR EQUATIONS.

1. An **Indeterminate Equation** was defined in Ch. XIII., § 1, Art. 1, as an equation which has an *indefinite* number of solutions; as $x + y = 5$.

An **Indeterminate System** is a system of equations which has an *indefinite* number of solutions.

Thus, if the system

$$x + y - z = 9,$$
$$2x - y + 7z = 33,$$

be solved for x and y, we obtain

$$x = 14 - 2z,\ y = 3z - 5.$$

In these values of x and y we may assign any value to z, and obtain corresponding values of x and y.

Evidently the number of solutions will be more limited if only ***positive integral*** values of the unknown numbers are admitted.

In this chapter we shall consider a simple method of solving in ***positive integers*** linear indeterminate equations and systems.

2. Ex. Solve $4x + 7y = 94$, in positive integers.

Solving for x, which has the smaller coefficient, we obtain

$$x = \frac{94 - 7y}{4} = 23 - y + \frac{2 - 3y}{4}, \qquad (1)$$

or

$$x - 23 + y = \frac{2 - 3y}{4}.$$

Since x and y are to be integers, $\frac{2 - 3y}{4}$ must be an integer; that is, y must have such a value that $2 - 3y$ shall be divisible by 4.

Let $$\frac{2 - 3y}{4} = m, \text{ an integer.}$$

Then $y = \frac{2 - 4m}{3}$, an inconvenient form from which to determine integral values of y. But since the expression $\frac{2 - 3y}{4}$ is to be an integer, any multiple of it will be an integer. We therefore multiply its numerator by the least number which will make the coefficient of y one more than a multiple of the denominator, *i.e.*, by 3. We then have

$$\frac{6-9y}{4}=1-2y+\frac{2-y}{4}, \text{ an integer.}$$

Therefore, as above, $\frac{2-y}{4}=m$, an integer.

Whence $\qquad y=2-4m. \qquad (2)$

Then, from (1) and (2), $\quad x=20+7m. \qquad (3)$

Any integral value of m will give to x and y integral values.

But since y is to be *positive*, $\quad m<1$;

and, since x is to be *positive*, $\quad m>-3$.

Therefore the only admissible values of m are 0, -1, -2.

When $m=0$, $\quad x=20,\quad y=2$;

$m=-1$, $\quad x=13,\quad y=6$;

$m=-2$, $\quad x=6,\quad y=10$.

In solving a system of *two* linear equations in *three* unknown numbers, we first eliminate one of the unknown numbers, and apply to the resulting equation the preceding method.

3. Pr. A party of 20 people, consisting of men, women, and children, pay a hotel bill of $67. Each man pays $5, each woman $4, and each child $1.50. How many of the company are men, how many women, and how many children?

Let x stand for the number of men,
y for the number of women,
z for the number of children.

Then, by the conditions of the problem,

$$x+y+z=20, \qquad (1)$$

$$5x+4y+\tfrac{3}{2}z=67. \qquad (2)$$

Eliminating z, we obtain $\quad 7x+5y=74.$

Whence $$y=\frac{74-7x}{5}=14-x+\frac{4-2x}{5}. \qquad (3)$$

Since x and y are to be integers, $\frac{4-2x}{5}$ must be an integer, and therefore

$$3\times\frac{4-2x}{5},\ =\frac{12-6x}{5},\ =2-x+\frac{2-x}{5}$$

must be an integer. Finally, let

$$\frac{2-x}{5}=m, \text{ an integer.}$$

Whence $x = 2 - 5m$. (4)

Then, from (3) and (4), $y = 12 + 7m$; (5)

and from (1), (4), and (5), $z = 6 - 2m$. (6)

Since x, y, and z are to be positive, we have:

from (4), $m < 1$; from (5), $m > -2$.

Therefore the only admissible values of m are -1, and 0.

When $m = 0$, $x = 2$, $y = 12$, $z = 6$;

$m = -1$, $x = 7$, $y = 5$, $z = 8$.

Consequently the company may have consisted of 2 men, 12 women, 6 children; or of 7 men, 5 women, and 8 children.

4. Not every linear indeterminate equation can be solved in positive integers.

The general form of such an equation is evidently $ax + by = c$, wherein a, b, and c are integers.

If a and b have a common factor, f say, then $a = fa'$, $b = fb'$, wherein a' and b' are integers. The equation may then be written

$$fa'x + fb'y = c, \text{ or } a'x + b'y = \frac{c}{f}.$$

Since a' and b' are integers, $\frac{c}{f}$ must be an integer, if x and y are to have integral values; that is, f must be also a factor of c. Therefore,

(i.) *The linear indeterminate equation $ax + by = c$ cannot be solved in positive integers if a and b have a common factor, which is not a factor of c.*

E.g., $2x - 4y = 5$ cannot be solved in positive integers.

We can infer at once that

(ii.) *If a and b are positive and c negative, the equation $ax + by = c$ cannot be solved in positive integers.*

For $ax + by$ would then be positive and could not be equal to c, a negative number.

E.g., $2x + 5y = -6$ cannot be solved in positive integers.

The case in which a and b are negative and c positive is evidently included in (ii.).

We therefore conclude that

(iii.) *The linear indeterminate equation $ax + by = c$ can be solved in positive integers only when a, b, and c are all positive, or when a and b have opposite signs; and when, in both cases, a and b do not have a common factor which is not also a factor of c.*

But even when the conditions given in (iii.) are satisfied, a solution in positive integers is not always possible.

Thus, the equation $7x + 9y = 15$ cannot be thus solved. For the least possible positive integral values of x and y are 1 and 1. But these give $7x + 9y = 16 \neq 15$.

EXERCISES.

Solve in positive integers:

1. $x + y = 10$. **2.** $2x + 3y = 25$. **3.** $5x + 7y = 52$.

4. $5x + 8y = 29$. **5.** $3x + 5y = 10$. **6.** $12x + 13y = 175$.

7. $25x + 15y = 215$. **8.** $5x + 13y = 229$. **9.** $34x + 89y = 407$.

10. $\begin{cases} x + 3y + 5z = 44, \\ 3x + 5y + 7z = 68. \end{cases}$ **11.** $\begin{cases} 8x + 3y - 2z = 8, \\ 7x - 2y - z = 8. \end{cases}$

Solve in least positive integers:

12. $91x - 221y = 0$. **13.** $3x - 5y = 1$. **14.** $17x - 11y = 86$.

15. $89x - 144y = 1$. **16.** $14x + 49y = 133$. **17.** $67x - 43y = 5$.

18. Divide 1000 into two parts so that one part shall be a multiple of 13, and the other a multiple of 53.

19. What positive integers when divided by 4 give a remainder 3, and when divided by 5 give a remainder 4?

20. Divide $\frac{79}{117}$ into two fractions with denominators 13 and 9 respectively.

21. A farmer received $16 for a number of turkeys and chickens. If he was paid $2 for each turkey and $.75 for each chicken, how many of each did he sell?

22. A gardener has fewer than 1000 trees. If he plants them in rows of 37 each, he will have 8 left; but if he plants them in a different number of rows of 43 each, he will have 11 left. How many trees has he?

23. A wheel with 17 teeth meshes in a wheel with 13 teeth. After how many revolutions of each wheel will each tooth occupy its original position?

24. A said to B: "If I had eight times as much money as I now have, and you had seven times as much money as you now have, and I were to give you $1, we should have equal amounts." How many dollars had each?

CHAPTER XVI.

EVOLUTION.

§ 1. DEFINITIONS AND PRINCIPLES.

1. A **qth Root** of a number or an expression is a number or an expression whose qth power is equal to the given one.

E.g., since $(+5)^2 = 25$ and $(-5)^2 = 25$, therefore $+5$ and -5 are *second* roots of 25. The statement, $+5$ and -5, is usually written ± 5.

Since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, therefore $a+b$ is a *third* root of $a^3 + 3a^2b + 3ab^2 + b^3$.

A *second* root of a number is usually called a *square* root; and a *third* root a *cube* root.

2. It follows from the definition of a root that a qth root of a number is one of q equal factors of the number.

Thus, either $+3$ or -3 is one of two equal factors of 9.

3. The **Radical Sign**, $\sqrt{}$, is used to denote a root, and is placed before the number or expression whose root is to be found.

The **Radicand** is the number or expression whose root is required.

The **Index** of a root is the number which indicates what root of the radicand is to be found, and is written over the radical sign. The index 2 is usually omitted.

E.g., $\sqrt[2]{9}$, or $\sqrt{9}$, denotes a second, or square root of 9; the radicand is 9, and the index is 2.

4. A vinculum is often used in connection with the radical sign to indicate what part of an expression following the sign is affected by it.

E.g., $\sqrt{}9+16$ means the sum of $\sqrt{}9$ and 16, while $\sqrt{9+16}$ means a square root of the sum $9+16$. Likewise $\sqrt[3]{}a^3 \times b^6$

means the product of $\sqrt[3]{a^3}$ and b^6, while $\sqrt[3]{a^3 \times b^6}$ means a cube root of a^3b^6.

Parentheses may be used instead of the vinculum in connection with the radical sign; as $\sqrt{}(9+16)$ for $\sqrt{9+16}$.

5. Like and Unlike Roots. — Two roots are said to be *like* or *unlike* according as their indices are equal or unequal, whether or not their radicands are equal.

E.g., $\sqrt[3]{a}$, $\sqrt[3]{b}$, are *like* roots; $\sqrt{a}$, $\sqrt[3]{a}$, are *unlike* roots.

6. In this chapter we shall consider only roots of numbers which are powers with exponents equal to or multiples of the indices of the required roots; as $\sqrt{16}, = \sqrt{4^2}$, $\sqrt[3]{a^3}$, $\sqrt[q]{a^{kq}}$.

An **Even Root** is one whose *index* is *even;* as $\sqrt{a^2}$, $\sqrt[4]{a^4}$, $\sqrt[2q]{a^{2q}}$.

An **Odd Root** is one whose *index* is *odd;* as $\sqrt[3]{8}$, $\sqrt[5]{a^{10}}$, $\sqrt[2q+1]{a^{2q+1}}$.

Number of Roots.

7. (i.) *A positive number has at least two even roots, equal and opposite;* i.e., *one positive and one negative.*

E.g., since $(\pm 4)^2 = 16$, $\sqrt{16} = \pm 4$; since $(\pm a)^4 = a^4$, $\sqrt[4]{a^4} = \pm a$. In general, since $(\pm a)^{2q} = a^{2q}$, $\sqrt[2q]{a^{2q}} = \pm a$.

(ii.) *A positive or a negative number has at least one odd root of the same sign as the number itself.*

E.g., since $(-3)^3 = -27$, $\sqrt[3]{-27} = -3$; since $2^5 = 32$, $\sqrt[5]{32} = 2$.

In general, since $(+a)^{2q+1} = +a^{2q+1}$, $\sqrt[2q+1]{+a^{2q+1}} = +a$.

Since $(-a)^{2q+1} = -a^{2q+1}$, $\sqrt[2q+1]{-a^{2q+1}} = -a$.

The principle enunciated in (ii.), when the radicand is negative, may also be stated as follows:

(iii.) *An odd root of a negative number is equal and opposite to a like root of a positive number which has the same absolute value.*

E.g., since $\sqrt[3]{-8} = -2$ and $-\sqrt[3]{8} = -2$,

therefore $\sqrt[3]{-8} = -\sqrt[3]{8}$;

since $\sqrt[2q+1]{-a^{2q+1}} = -a$ and $-\sqrt[2q+1]{a^{2q+1}} = -a$,

therefore $\sqrt[2q+1]{-a^{2q+1}} = -\sqrt[2q+1]{a^{2q+1}}$.

Consequently, to find an odd root of a negative number, find a like root of the positive number which has the same absolute value, and prefix the negative sign to this root.

(iv.) *Since* $0^n = 0$, *therefore* $\sqrt[n]{0} = 0$.

(v.) *Since* $(+4)^2 = +16$ *and* $(-4)^2 = +16$, *there is no number, with which we are as yet familiar, whose square is* -16. Consequently $\sqrt{-16}$ cannot be expressed in terms of the numbers as yet used in this book.

In general, since $(\pm a)^{2n} = +a^{2n}$, we cannot express $\sqrt[2n]{-a^{2n}}$ in terms of numbers hitherto used.

8. It was shown in Art. 7 that a positive number, which is the qth power of a number, has *at least one qth root,* and when q is even *at least two;* also that any negative number, which is an odd power of a negative number, has *at least one odd root.*

It will be proved in Chapters XXI. and XXII. that any number has *two square* roots, *three cube* roots, *four fourth* roots; and in Part II., Text-Book of Algebra, that, in general, any number has q qth roots.

Principal Roots.

9. The **Principal Root** of a *positive* number is its one *positive* root.

E.g., 3 is the principal square root of 9.

The **Principal Odd Root** of a *negative* number is its one *negative* root.

E.g., -2 is the principal cube root of -8.

10. $\sqrt{4^2} = \sqrt{16} = \pm 4$, *if other than the principal root be admitted,* and $(\sqrt{4})^2 = (\pm 2)^2 = 4$; therefore, $\sqrt{4^2} = (\sqrt{4})^2$, only for the principal square root.

In general $\sqrt[q]{a^q} = (\sqrt[q]{a})^q$ *is true only for the principal qth root.*

In subsequent work the radical sign will be understood to denote only the principal root, *unless the contrary is stated.*

E.g., $\sqrt{9} = 3,\ -\sqrt{16} = -4,\ \sqrt[3]{-27} = -3.$

EXERCISES I.

Write

1. Two square roots of 49.
2. Two fourth roots of 81.
3. Two sixth roots of 64.
4. Two square roots of 5^{2n}.

Write one cube root of

5. 64.
6. -125.
7. 1000.
8. $-a^{3n}$.

Find the value of the indicated principal root of each of the following numbers:

9. $\sqrt{256}$.
10. $\sqrt[3]{216}$.
11. $\sqrt[3]{-512}$.
12. $\sqrt{625}$.
13. $\sqrt[3]{-729}$.
14. $\sqrt[4]{1296}$.
15. $\sqrt[5]{243}$.
16. $\sqrt[6]{64}$.

From the definition of a root, express a as a root of the second member of each of the following equations:

17. $a^3 = b$.
18. $a^4 = b^8$.
19. $a^5 = b^7$.
20. $a^n = b^m$.

21–24. In Exx. 17–20, express b as a root of the first member of each of the equations.

Evolution.

11. Evolution is the process of finding any required root of a given number or expression.

Since even roots of negative numbers are not considered in this chapter, and since, by Art. 7 (iii.), an odd root of a negative number can be found from the like root of a positive number, we shall give now only methods for finding principal roots of positive numbers and expressions.

In the following principles the radicands are limited to positive values, and the roots to principal roots.

Principles of Roots.

12. *The like principal roots of equal numbers or expressions are equal.*

If $a = b$, then $\sqrt[q]{a} = \sqrt[q]{b}$.

This principle follows directly from axioms (i.) and (iii.).

13. The process of evolution depends upon the following principles:

(i.) *The principal root of a power of a number is equal to the same power of the like principal root of the number, and conversely;* or, stated symbolically,

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

In particular, $$\sqrt[q]{a^q} = (\sqrt[q]{a})^q.$$

E.g., $\sqrt[3]{8^5} = (\sqrt[3]{8})^5 = 2^5 = 32$; $\sqrt[5]{32^5} = (\sqrt[5]{32})^5 = 32.$

(ii.) *The principal root of a power is obtained by dividing the exponent of the power by the index of the root;* or, stated symbolically,

$$\sqrt[q]{a^{kq}} = a^{\frac{kq}{q}} = a^k.$$

E.g., $$\sqrt[3]{a^6} = a^{\frac{6}{3}} = a^2.$$

(iii.) *The principal root of a product of two or more factors is equal to the product of the like principal roots of the factors, and conversely;* or, stated symbolically,

$$\sqrt[q]{(ab)} = \sqrt[q]{a} \times \sqrt[q]{b}, \text{ and } \sqrt[q]{a} \times \sqrt[q]{b} = \sqrt[q]{(ab)}.$$

E.g., $\sqrt{(16 \times 25)} = \sqrt{16} \times \sqrt{25} = 4 \times 5 = 20;$

$$\sqrt[3]{(8\,a^3b^6)} = \sqrt[3]{8} \times \sqrt[3]{a^3} \times \sqrt[3]{b^6} = 2 \times a \times b^2 = 2\,ab^2.$$

(iv.) *The principal root of a quotient of two numbers is equal to the quotient of the like principal roots of the numbers, and conversely;* or, stated symbolically,

$$\sqrt[q]{\frac{a}{b}} = \frac{\sqrt[q]{a}}{\sqrt[q]{b}}, \text{ and } \frac{\sqrt[q]{a}}{\sqrt[q]{b}} = \sqrt[q]{\frac{a}{b}}.$$

E.g., $\sqrt{\frac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4}$; $\sqrt[3]{\frac{27\,a^3}{b^6}} = \frac{\sqrt[3]{(27\,a^3)}}{\sqrt[3]{b^6}} = \frac{3\,a}{b^2}.$

(v.) *The principal root of the principal root of a number is equal to that principal root of the number whose index is equal to the product of the indices of the given roots, and conversely;* or, stated symbolically,

$$\sqrt[q]{\sqrt[r]{a}} = \sqrt[qr]{a}, \text{ and } \sqrt[qr]{a} = \sqrt[q]{\sqrt[r]{a}} = \sqrt[r]{\sqrt[q]{a}}.$$

E.g., $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64} = 2$; $\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4.$

The proofs follow:

(i.) Let the qth root of a be denoted by R, or $\sqrt[q]{a} = R$.

Then $(\sqrt[q]{a})^q = R^q$, by Ch. II., § 6, Art. 7,

or $a = R^q$, since $(\sqrt[q]{a})^q = a$, by definition of a root;

and $a^p = (R^q)^p$, by Ch. II., § 6, Art. 7,

or $a^p = (R^p)^q$, since $(R^q)^p = (R^p)^q$.

Whence $\sqrt[q]{a^p} = \sqrt[q]{(R^p)^q}$, by Art. 12,

$= R^p$, by Art. 10.

Substituting $\sqrt[q]{a}$ for R in the last equation, we have

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

(ii.) Let the q root of a^{kq} be denoted by R, or $R = \sqrt[q]{a^{kq}}$.

Then $R^q = (\sqrt[q]{a^{kq}})^q$, by Ch. II., § 6, Art. 7,

$= a^{kq} = (a^k)^q$.

Whence $\sqrt[q]{R^q} = \sqrt[q]{(a^k)^q}$, or $R = a^k$, by Art. 10.

Substituting $\sqrt[q]{a^{kq}}$ for R in the last equation, we have

$$\sqrt[q]{a^{kq}} = a^k = a^{\frac{kq}{q}}.$$

(iii.) Let $\sqrt[q]{a} = R$, and $\sqrt[q]{b} = R_1$.

Then $(\sqrt[q]{a})^q = R^q$, and $(\sqrt[q]{b})^q = R_1{}^q$, by Ch. II., § 6, Art. 7,

or $a = R^q$, and $b = R_1{}^q$.

Therefore $ab = R^qR_1{}^q = (RR_1)^q$.

Whence $\sqrt[q]{(ab)} = RR_1$, by Arts. 10 and 12.

Substituting $\sqrt[q]{a}$ for R, and $\sqrt[q]{b}$ for R_1 in the last equation, we have

$$\sqrt[q]{(ab)} = \sqrt[q]{a}\sqrt[q]{b}.$$

In like manner, the principle can be proved for the qth root of a product of any number of factors.

In like manner, (iv.) and (v.) can be proved.

§ 2. ROOTS OF MONOMIALS.

1. The principal (*positive*) root of a positive number or expression can be found by applying the principles of § 1, Art. 13.

The *negative even* root of a positive number or expression is found by prefixing the negative sign to its principal root.

The *negative odd* root of a negative number or expression is ound by prefixing the negative sign to the principal root of the adicand taken positively.

Ex. 1. $\sqrt{(16a^2b^4)} = \sqrt{16} \times \sqrt{a^2} \times \sqrt{b^4}$

$= 4ab^2$, the principal square root.

Therefore $\pm\sqrt{(16a^2b^4)} = \pm 4ab^2$.

In the following examples we shall give only the *positive* even roots.

Ex. 2. $\sqrt[3]{(-27x^3y^6z^9)} = \sqrt[3]{-27} \times \sqrt[3]{x^3} \times \sqrt[3]{y^6} \times \sqrt[3]{z^9}$

$= -3xy^2z^3$.

Ex. 3. $\sqrt[4]{\dfrac{16a^8b^{12}}{625c^{16}}} = \dfrac{\sqrt[4]{(16a^8b^{12})}}{\sqrt[4]{(625c^{16})}} = \dfrac{\sqrt[4]{16} \times \sqrt[4]{a^8} \times \sqrt[4]{b^{12}}}{\sqrt[4]{625} \times \sqrt[4]{c^{16}}} = \dfrac{2a^2b^3}{5c^4}$.

Ex. 4. $\sqrt[3]{(27a^3)^2} = (\sqrt[3]{27a^3})^2 = (3a)^2 = 9a^2$.

The work of the last example is much simplified by taking the power of the root instead of the root of the power.

It is frequently necessary to modify the form of the radicand before finding the indicated root.

Ex. 5. $\sqrt{(35 \times 7 \times 20)} = \sqrt{(7 \times 5 \times 7 \times 4 \times 5)}$

$= \sqrt{(7^2 \times 5^2 \times 4)} = 7 \times 5 \times 2 = 70$.

Ex. 6. $\sqrt[2n+1]{(-a^{4n+2}b^{4n^2-1})} = -\sqrt[2n+1]{a^{4n+2}} \times \sqrt[2n+1]{b^{4n^2-1}} = -a^2b^{2n-1}$.

EXERCISES II.

Simplify each of the following expressions:

1. $\sqrt{(16a^2b^8)}$.
2. $\sqrt{(36a^4b^{10}c^6)}$.
3. $\sqrt[3]{(8a^3b^6)}$.
4. $\sqrt[3]{(-64a^9b^{12}x^{15})}$.
5. $\sqrt[4]{(8a^{10}b^4 \times 2a^2b^4)}$.
6. $\sqrt[5]{(-a^{20}x^{35})}$.
7. $\sqrt{(3ax^{3n} \times 27a^3x^{3n})}$.
8. $\sqrt[3]{(9a^4x^{14}y^{2n} \times 3a^2x^{10}y^n)}$.
9. $\sqrt[4]{(5\tfrac{1}{16}x^{4n}y^{8n-12})}$.
10. $\sqrt[6]{(64x^{10}y^{11}z^2 \times x^2y^7z^4)}$.
11. $\sqrt{[81a^4(a^2+x^2)^6]}$.
12. $\sqrt{(6\tfrac{1}{4}a^6b^4c^{4p-6})}$.
13. $\sqrt[2n+1]{(-a^{2n+1}b^{6n+3})}$.
14. $\sqrt[3]{[3\tfrac{3}{8}x^{3n-9}(x-1)^9]}$.
15. $\sqrt{\dfrac{49a^{10}}{b^4c^6}}$.
16. $\sqrt[3]{-\dfrac{a^{21}x^{15}}{343}}$.
17. $\sqrt[3]{\dfrac{27a^3b^6}{64x^9y^{12}}}$.
18. $\sqrt{\dfrac{9a^6b^{4m}}{c^{10}d^{2n}}}$.
19. $\sqrt[4]{\dfrac{625x^4y^{12}}{a^8b^{16}}}$.
20. $\sqrt[3]{\dfrac{.064a^{12}}{b^3x^{15n}}}$.

21. $\sqrt[5]{\dfrac{a^5x^{20}}{(a-x)^{10}}}$.　　　**22.** $\sqrt[6]{\dfrac{a^{12}b^{24}c^6}{64\,x^{18}y^{42}}}$.

23. $\sqrt{}\sqrt[3]{}(49^3 \times 64^3)$.　　**24.** $\sqrt[3]{}\sqrt{}(5^6x^{12}y^6)$.　　**25.** $\sqrt[3]{}\sqrt{}(4096\,a^{12}b^{18})$.

26. $\sqrt{}\sqrt{}(16\,a^{4n-4})$.　　**27.** $\sqrt[3]{}\sqrt[n]{}(27^na^{3n}b^{6n})^2$.　　**28.** $\sqrt[5]{}\sqrt[3]{}(-3^{15}a^{30}b^{15n})$

29. $\sqrt{}(100\,a^{2n}b^4)^3$.　　**30.** $\sqrt[3]{}(\frac{27}{64})^2$.　　**31.** $\sqrt[3]{}(8\,a^6b^9x^{12})^5$.

§ 3. SQUARE ROOTS OF MULTINOMIALS.

1. The square root of a trinomial which is the square of a binomial and the square roots of certain multinomials can be found by inspection (Ch. VIII., § 1, Art. 9).

2. Since $$(a+b)^2 = a^2 + 2\,ab + b^2,$$
we have $$\sqrt{}(a^2 + 2\,ab + b^2) = a + b.$$

From this identity we infer:

(i.) *The first term of the root is the square root of the first term of the trinomial;* i.e., $a = \sqrt{}a^2$.

(ii.) *If the square of the first term of the root be subtracted from the trinomial, the remainder will be*
$$2\,ab + b^2, \;= (2\,a + b)\,b.$$

Twice the first term of the root, $2\,a$, is called the *Trial Divisor.*

(iii.) *The second term of the root is obtained by dividing the first term of the remainder by the trial divisor;* i.e., $b = \dfrac{2\,ab}{2\,a}$.

(iv.) *If twice the first term of the root plus the second term,* $2\,a + b$ (*the complete divisor*), *be multiplied by the second term,* b, *and the product be subtracted from the first remainder, the second remainder will be* 0.

The work may be arranged as follows:

$$\begin{array}{r|ll}
a^2 + 2\,ab + b^2 & \underline{a + b} & \\
\underline{a^2\qquad\quad} & 2\,a & \text{trial divisor} \\
2\,ab & 2\,ab \div 2\,a = b, & \text{second term of root} \\
 & 2\,a + b & \text{complete divisor} \\
\underline{2\,ab + b^2} & = (2\,a + b)\,b &
\end{array}$$

Ex. Find the square root of $4x^4 - 12x^2y + 9y^2$.

The work, arranged as above, writing only the trial and the complete divisor, is:

$$\begin{array}{l|l}
4x^4 - 12x^2y + 9y^2 & 2x^2 - 3y \\ \cline{2-2}
4x^4 & 4x^2 \\ \cline{1-1}
\quad -12x^2y & \\
\quad -12x^2y + 9y^2 & 4x^2 - 3y \\ \hline
\end{array}$$

3. When the multinomial is the square of a trinomial, the process of finding the root is an extension of the method of Art. 2.

The multinomial whose root is required should be arranged to powers of a letter of arrangement.

Since
$$(a + b + c)^2 = (a + b)^2 + 2(a + b)c + c^2$$
$$= (a^2 + 2ab + b^2) + 2ac + 2bc + c^2,$$

we have $\sqrt{[(a^2 + 2ab + b^2) + (2a + 2b + c)c]} = a + b + c$.

The first two terms of the root are found by inspection, or by the method of Art. 2. The work may be arranged as follows:

$$\begin{array}{l|ll}
a^2+2ab+b^2+2ac+2bc+c^2 & a+b+c & \text{required root} \\ \cline{2-2}
a^2 & 2a & \text{trial divisor} \\ \cline{1-1}
\quad 2ab & 2ab \div 2a = b, & \text{second term of root} \\
\quad 2ab+b^2 & 2a+b & \text{complete divisor of first stage} \\ \cline{1-1}
\qquad 2ac & 2ac \div 2a = c, & \text{third term of root} \\
 & 2a+2b+c, & \text{complete divisor of second stage} \\
\qquad 2ac+2bc+c^2 & & \\ \hline
\end{array}$$

Observe that $2ac$ is the first term of the remainder after subtracting $(a + b)^2 = a^2 + 2ab + b^2$. For, in finding the first two terms of the root we first subtracted a^2 and then $2ab + b^2$. Notice also that the complete divisor at any stage is twice the part of the root already found, plus the term last found.

Ex. 1. Find the square root of

$$4x^4 - 12x^3 + 29x^2 - 30x + 25.$$

The work follows:

$$\begin{array}{r|l}
4x^4 - 12x^3 + 29x^2 - 30x + 25 & \underline{2x^2 - 3x + 5} \\
\underline{4x^4} \qquad\qquad\qquad\qquad\qquad\quad & 4x^2 \\
-12x^3 \qquad\qquad\qquad\qquad & \\
\underline{-12x^3 + 9x^2} \qquad\qquad\quad & \underline{4x^2 - 3x} \\
20x^2 \qquad\qquad\quad & \\
\underline{20x^2 - 30x + 25} & \underline{4x^2 - 6x + 5}
\end{array}$$

Only the trial divisor and the complete divisor of each stage are written, the other steps being performed mentally.

4. The preceding method can be extended to find square roots which are multinomials of any number of terms.

The work consists of repetitions of the following steps:

After one or more terms of the root have been found, obtain each succeeding term, by dividing the first term of the remainder at that stage by twice the first term of the root.

Find the next remainder by subtracting from the last remainder the expression $(2a + b)b$, *wherein* a *stands for the part of the root already found, and* b *for the term last found.*

EXERCISES III.

Find the square root of each of the following expressions:

1. $x^4 - 4x^3 + 8x + 4.$
2. $4m^4 - 4m^3 + 5m^2 - 2m + 1.$
3. $x^4 - 2x^3 + 3x^2 - 2x + 1.$
4. $4x^4 + 12x^3 + 5x^2 - 6x + 1.$
5. $9x^4 + 12x^3 - 26x^2 - 20x + 25.$
6. $4x^4 - 28x^3 + 51x^2 - 7x + \frac{1}{4}.$
7. $x^4y^4 - 4x^3y^3 + 6x^2y^2 - 4xy + 1.$
8. $\frac{1}{9}x^4 + \frac{4}{3}x^3y + 2x^2y^2 - 12xy^3 + 9y^4.$
9. $x^4 - 6ax^3 + 13a^2x^2 - 12a^3x + 4a^4.$
10. $4a^2 + 9b^2 + 16c^2 - 12ab + 16ac - 24bc.$
11. $49x^8 + 42x^6 - 19x^4 - 12x^2 + 4.$
12. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4.$
13. $a^2 + 4a^3 + 4a^2 + 2a + 4 + \frac{1}{a^2}.$

14. $\frac{x^4}{y^4} - \frac{4x^3}{y} + 4x^2y^2 + 6x - 12y^3 + 9\frac{y^4}{x^2}$.

15. $x^4 + \frac{2x^3}{a} + \frac{x^2}{a^2} + 2ax + 2 + \frac{a^2}{x^2}$. **16.** $1 + 2x - x^2 + 3x^4 - 2x^5 + x^6$.

17. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$.

18. $1 - 4a + 64a^6 - 64a^5 - 32a^3 + 48a^4 + 12a^2$.

19. $4a^6 + 17a^2 - 22a^3 + 13a^4 - 24a - 4a^5 + 16$.

20. $9x^6 + 6x^5y + 43x^4y^2 + 2x^3y^3 + 45x^2y^4 - 28xy^5 + 4y^6$.

21. $x^4 + 4x^3 + 6x^2 + 5x + 5 + \frac{5}{x} + \frac{9}{4x^2} + \frac{1}{x^3} + \frac{1}{x^4}$.

22. $a^{2m}x^{2n} + 10a^{2m-2}x^{2n+1} - 6a^{m+1}x^{n+1} + 25a^{2m-4}x^{2n+2} - 30a^{m-1}x^{n+2}$
$+ 9a^2x^2$.

§ 4. CUBE ROOTS OF MULTINOMIALS.

1. The process of finding the cube root of a multinomial is the inverse of the process of cubing the multinomial.

Since
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + (3a^2 + 3ab + b^2)b, \qquad (1)$$
we have
$$\sqrt[3]{(a^3 + 3a^2b + 3ab^2 + b^3)} = a + b. \qquad (2)$$

From the identity (2), we infer:

(i.) *The first term of the root is the cube root of the first term of the multinomial;* i.e., $a = \sqrt[3]{a^3}$.

(ii.) *If the cube of the first term of the root be subtracted from the multinomial, the remainder will be*

$$3a^2b + 3ab^2 + b^3, = (3a^2 + 3ab + b^2)b.$$

Three times the square of the first term of the root, $3a^2$, is called the *Trial Divisor.*

(iii.) *The second term of the root is obtained by dividing the first term of the remainder by the trial divisor;* i.e., $b = \frac{3a^2b}{3a^2}$.

(iv.) *If the sum* $3a^2 + 3ab + b^2$, *the complete divisor, be multiplied by the second term of the root, and this product be subtracted from the first remainder, the second remainder will be* 0.

The work may be arranged as follows:

$$\begin{array}{r|lll}
a^3+3a^2b+3ab^2+b^3 & \underline{a+b} & & \\
\underline{a^3\qquad\quad} & 3a^2 & \text{trial divisor} & (1) \\
3a^2b\qquad & 3a^2b\div 3a^2=b, & \text{second term of root} & (2) \\
 & 3a^2+3ab+b^2, & \text{complete divisor} & (3) \\
\underline{3a^2b+3ab^2+b^3} & =(3a^2+3ab+b^2)\times b & & (4)
\end{array}$$

Ex. 1. Find the cube root of $27x^3+54x^2y+36xy^2+8y^3$.

The work, arranged as above, is:

$$\begin{array}{r|lll}
27x^3+54x^2y+36xy^2+8y^3 & \underline{3x+2y} & & \\
\underline{27x^3\qquad\qquad\qquad} & 3(3x)^2=27x^2, & \text{trial divisor} & (1) \\
54x^2y\qquad\qquad & 54x^2y\div 27x^2=2y, & \text{second term of root} & (2) \\
 & 3(3x)^2+3(3x)(2y)+(2y)^2=27x^2 & & \\
 & \quad +18xy+4y^2, & \text{complete divisor} & (3) \\
\underline{54x^2y+36xy^2+8y^3} & =(27x^2+18xy+4y^2)(2y) & & (4)
\end{array}$$

2. The preceding method can be extended to find cube roots which are multinomials of any number of terms, as the method of finding square roots was extended. The work consists of repetitions of the following steps:

After one or more terms of the root have been found, obtain each succeeding term by dividing the first term of the remainder at that stage by three times the square of the first term of the root.

Find the next remainder by subtracting from the last remainder the expression $(3a^2+3ab+b^2)b$, *wherein* a *stands for the part of the root already found, and* b *for the term last found.*

The given multinomial should be arranged to powers of a letter of arrangement.

Ex.

$$\begin{array}{r|l}
27-27x+90x^2-55x^3+90x^4-27x^5+27x^6 & \underline{3-x+3x^2} \\
\underline{27\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & 3(3)^2=27 \\
-27x\qquad\qquad\qquad\qquad\qquad\qquad & \\
\underline{-27x+9x^2-x^3\qquad\qquad\qquad\qquad} & \underline{3(3)^2+3(3)(-x)+(-x)^2=27-9x+x^2} \\
81x^2-54x^3\qquad\qquad\qquad\qquad & 3(3-x)^2+3(3-x)(3x^2)+(3x^2)^2= \\
\underline{81x^2-54x^3+90x^4-27x^5+27x^6} & \underline{27-18x+30x^2-9x^3+9x^4}
\end{array}$$

EXERCISES IV.

Find the cube root of each of the following expressions:

1. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$
2. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$
3. $156a^4 - 144a^5 - 99a^3 + 64a^6 + 39a^2 - 9a + 1.$
4. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$
5. $1 - 6x + 9x^2 + 4x^3 - 9x^4 - 6x^5 - x^6.$
6. $8x^3 - 12x^2 + 12x - 7 + \frac{3}{x} - \frac{3}{4x^2} + \frac{1}{8x^3}.$
7. $27a^6x^6 + 54a^5x^5 + 9a^4x^4 - 28a^3x^3 - 3a^2x^2 + 6ax - 1.$
8. $8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6.$
9. $x^3 + 3x^7 - 9x^{11} - 27x^{15} - 6x^5 - 54x^{13} + 28x^9.$
10. $108a^5 - 48a^4 + 8a^3 + 54a^7 - 12a^8 + a^9 - 112a^6.$
11. $8a^6 - 48a^5x + 60a^4x^2 - 27x^6 - 108ax^5 - 90a^2x^4 + 80a^3x^3.$
12. $1 + 3x - 8x^3 - 6x^4 + 6x^5 + 8x^6 - 3x^8 - x^9.$
13. $\frac{125y^6}{x^6} - \frac{150y^5}{x^5} - \frac{165y^4}{x^4} + \frac{172y^3}{x^3} + \frac{99y^2}{x^2} - \frac{54y}{x} - 27.$
14. $64x^{3n} - 144x^{3n-1} + 12x^{3n-2} + 117x^{3n-3} - 6x^{3n-4} - 36x^{3n-5} - 8x^{3n-6}.$

§ 5. HIGHER ROOTS.

1. Since $\sqrt[4]{N} = \sqrt{\sqrt{N}}$, wherein N stands for any multinomial, the fourth root is most easily found as the square root of the square root of the given multinomial.

In like manner, since $\sqrt[6]{N} = \sqrt[3]{\sqrt{N}}$, the sixth root can be found as the cube root of the square root of the given multinomial. And so on for any root whose index can be factored.

2. The process of finding the nth root of a multinomial is the inverse of raising a multinomial to the nth power.

The method can be derived from the expression for $(a + b)^n$, which will be given in Ch. XXVIII.

EXERCISES V.

Find the fourth root of each of the following expressions:

1. $x^8 + 4x^6 + 6x^4 + 4x^2 + 1.$
2. $a^8 + 4a^7b + 10a^6b^2 + 16a^5b^3 + 19a^4b^4 + 16a^3b^5 + 10a^2b^6 + 4ab^7 + b^8.$

3. $16x^8 - 160x^7 + 408x^6 + 440x^5 - 2111x^4 - 1320x^3 + 3672x^2 + 4320x$
$+ 1296$.

4. $625x^8 + 5500x^7 + 17150x^6 + 20020x^5 + 721x^4 - 8008x^3 + 2744x^2$
$- 352x + 16$.

Find the sixth roots of each of the following expressions :

5. $64x^{12} - 192x^{10} + 240x^8 - 160x^6 + 60x^4 - 12x^2 + 1$.

6. $a^{12} + 6a^{11}b + 21a^{10}b^2 + 50a^9b^3 + 90a^8b^4 + 126a^7b^5 + 141a^6b^6$
$+ 126a^5b^7 + 90a^4b^8 + 50a^3b^9 + 21a^2b^{10} + 6ab^{11} + b^{12}$.

§ 6. ROOTS OF ARITHMETICAL NUMBERS.

Square Roots.

1. Since the squares of the numbers 1, 2, 3, ⋯, 9, 10, are 1, 4, 9, ⋯, 81, 100, respectively, the square root of an integer of *one or two* digits is a number of *one* digit.

Since the squares of the numbers 10, 11, ⋯, 100, are 100, 121, ⋯, 10000, the square root of an integer of *three or four* digits is a number of *two* digits.

In general, the square root of any integer of $2n - 1$ or $2n$ digits is a number of n digits.

Therefore, *to find the number of digits in the square root of a given integer, we first mark off the digits from right to left in groups of two. The number of digits in the square root will be equal to the number of groups, counting any one digit remaining on the left as a group.*

2. The method of finding square roots of numbers is then derived from the identity

$$(a + b)^2 = a^2 + (2a + b)b, \qquad (1)$$

wherein a denotes *tens*, and b denotes *units*, if the square root be a number of two digits.

Ex. 1. Find the square root of 1296.

We see that the root is a number of *two* digits, since the given number divides into *two* groups. The digit in the *tens'* place is 3, the square root of 9, the square next less than 12. Therefore, in the identity (1), a denotes 3 *tens*, or 30.

The work then proceeds as follows:

$$\begin{array}{r|ll}
 & a+b & \\
12'96 & 30+6=36 & \\
\underline{9\ 00} & 2a=60, \quad \text{trial divisor} & (1) \\
3\ 96 & (2ab+b^2)\div 2a = 396 \div 60 = 6+ & (2) \\
3\ 96 & = (2a+b)\times b = (60+6)\times 6 & (3)
\end{array}$$

The first remainder, 396, is equal to $2ab+b^2$, and cannot be separated into the sum of two terms, one of which is $2ab$. We cannot, therefore, determine b by dividing $2ab$ by $2a$, as in finding square roots of algebraic expressions.

Consequently step (2) *suggests* the value of b but does not definitely determine it. As a rule we take the integral part of the quotient, 6 in the above example, and test that value by step (3).

This method may be extended to find roots which contain any number of digits. At any stage of the work a stands for the part of the root already found, and b for the digit to be found.

Ex. **2**. Find the square root of 51529.

The root is a number of *three* digits, since the given number divides into *three* groups. The digit in the *hundreds'* place is 2, the square root of 4, the square next less than 5. Therefore in the identity (1), a denotes 2 *hundreds*, or 200, in the first stage of the work.

The work then proceeds as follows:

$$\begin{array}{r|ll}
5'15'29 & 200+20+7=227 & \\
\underline{4\ 00\ 00} & 2a=400, \quad \text{trial divisor} & (1) \\
1\ 15\ 29 & (2ab+b^2)\div 2a = 11529 \div 400 = 20+ & (2) \\
\underline{84\ 00} & = (2a+b)b = (400+20)\times 20 & (3) \\
31\ 29 & (2ab+b^2)\div 2a = 3129 \div 440 = 7+ & (4) \\
\underline{31\ 29} & = (2a+b)b = (440+7)\times 7 & (5)
\end{array}$$

In the second stage of the work, a stands for the part of the root already found, 220, and b for the next figure of the root. In practice the work may be arranged more compactly, omitting unnecessary ciphers, and in each remainder writing only the next group of figures. Thus:

$$\begin{array}{r|l}
5'15'29 & \underline{227} \\
\underline{4} & \\
1\ 15 & 11 \div 4 = 2 + \qquad\qquad (2) \\
\underline{84} & \underline{42} \\
31\ 29 & 312 \div 44 = 7 + \qquad (4) \\
\underline{31\ 29} & \underline{447}
\end{array}$$

Observe that the trial divisor at any stage is twice the part of the root already found, as in (2) and (4).

The abbreviated work in the last example illustrates the following method:

After one or more figures of the root have been found, obtain the next figure of the root by dividing the remainder at that stage (omitting the last figure), by the trial divisor at that stage.

See lines (2) and (4).

Annex this quotient to the part of the root already found, and also to the trial divisor to form the complete divisor.

Find the next remainder by subtracting from the last remainder the product of the complete divisor and the figure of the root last found.

3. Since the number of decimal places in the square of a decimal fraction is twice the number of decimal places in the fraction, the number of decimal places in the square root of a decimal fraction is one-half the number of decimal places in the fraction.

Consequently, in finding the square root of a decimal fraction, the decimal places are divided into groups of two from the decimal point to the right, and the integral places from the decimal point to the left as before.

Ex.

$$\begin{array}{r|l}
14'46.28'09 & \underline{38.03} \\
\underline{9} & \\
5\ 46 & \\
\underline{5\ 44} & \underline{68} \\
2.28\ 09 & \\
\underline{2.28\ 09} & \underline{76.03}
\end{array}$$

In finding the second figure of the root, we have $\frac{54}{6} = 9$; but $59 \times 9 = 621$, which is greater than 546, from which it is to be subtracted. Hence we take the next less figure 8.

Cube Roots.

4. Since the cubes of the numbers 1, 2, 3, ⋯, 9, 10 are 1, 8, 27, ⋯, 729, 1000, respectively, the cube root of any integer of *one*, *two*, or *three* digits is a number of *one* digit. *The cube roots of such numbers can be found only by inspection.*

Since the cubes of 10, 11, ⋯, 100 are 1000, 1331, ⋯, 1000000, respectively, the cube root of any integer of *four*, *five*, or *six* digits is a number of *two* digits.

In general, the cube root of any integer of $3n-2$, $3n-1$, or $3n$ digits is a number of n digits.

Therefore, to find the number of digits in the cube root of a given integer, we first mark off the digits from right to left in groups of *three*. The number of digits in the cube root will be equal to the number of groups, counting one or two digits remaining on the left as a group.

5. The method of finding cube roots of numbers is derived from the identity

$$(a+b)^3 = a^3 + (3a^2 + 3ab + b^2)b,$$

wherein a denotes *tens*, and b denotes *units*, if the cube root is a number of two digits.

Ex. Find the cube root of 59319.

The digits in the *tens'* place of the root is 3, the cube root of 27, the cube next less than 59. Therefore in identity (1), a denotes 3 *tens* or 30. The work may be arranged as follows:

59′ 319	$a+b$	
27 000	$30+9$	
32 319	$3a^2 = 3(30)^2 = 2700$	(1)
	$(3a^2b + 3ab^2 + b^3) \div 3a^2 = 32319 \div 2700 = 9+$	(2)
	$3a^2 = 3(30)^2 = 2700$	
	$3ab = 3(30)9 = 810$	
	$b^2 = 9^2 = 81$	
32 319	$= (3a^2 + 3ab + b^2) \times b = 3591 \times 9$	(3)

As in finding square roots of numbers, step (2) *suggests* the value of b, but does not definitely determine it. If the value of b makes $(3a^2 + 3ab + b^2) \times b$ greater than the number from which it is to be subtracted, we must try the next less number.

In practice the work may be arranged more compactly, omitting unnecessary ciphers, and in each remainder writing only the next group of figures; thus

$$\begin{array}{r|r l}
59'319 & 39 & \\ \cline{2-2}
27 & & \\ \cline{1-1}
32\ 319 & 2700 & \quad (1) \\
 & 810 & \quad (2) \\
 & 81 & \quad (3) \\ \cline{2-2}
32\ 319 & 3591 & \\ \hline
\end{array}$$

6. The preceding method may be extended to find roots that contain any number of digits.

At any stage of the work a stands for the part of the root already found, and b for the digit to be found.

The method consists of a repetition of the following steps:

The trial divisor at any stage is three times the square of the part of the root already found; as 27 in the preceding example.

After one or more figures of the root have been found obtain the next figure of the root by dividing the remainder at that stage (omitting the last two figures) by the trial divisor. In the last example, $9 + = 323 \div 27$.

Annex this quotient to the part of the root already found.

Add to the trial divisor (with two ciphers annexed) three times the product of the part of the root already found (with one cipher annexed) and the figure of the root just found, and also the square of the figure of the root just found. The sum is called the complete divisor.

Find the next remainder by subtracting from the last remainder the product of the complete divisor and the figure of the root last found.

7. Evidently, in finding the cube root of a decimal fraction the decimal places are divided into groups of *three* figures from the decimal point to the right, and the integral places from the decimal point to the left as before.

Ex.

11'089.567	22.3
8	1200
3 089	120
	4
2 648	1324
441.567	1452.00
	19.80
	.09
441.567	1471.89

EXERCISES VI.

Find the square root of each of the following numbers:

1. 196. **2.** 841. **3.** 1296. **4.** 65.61. **5.** 7396.

6. 3481. **7.** 667489. **8.** 170569. **9.** 1664.64.

10. 582169. **11.** 1.737124. **12.** 556.0164. **13.** .00099225.

Find the cube root of each of the following numbers:

14. 2744. **15.** 39304. **16.** 110.592. **17.** 328509.

18. 1.191016. **19.** 74088000. **20.** 340068392. **21.** 426.957777.

22. 584067.412279. **23.** 375601280.458951. **24.** .041063625.

Find the value of each of the following indicated roots:

25. $\sqrt[4]{279841}$. **26.** $\sqrt[6]{3010936384}$. **27.** $\sqrt[4]{164204746.7776}$.

CHAPTER XVII.

INEQUALITIES.

1. One number is greater or less than a second number according as the remainder of subtracting the second number from the first is positive or negative. Thus,

$a > b$, when $a - b$ is *positive*, *i.e.*, when $a - b > 0$.

$a < b$, when $a - b$ is *negative*, *i.e.*, when $a - b < 0$.

2. An **Inequality** is a statement that two numbers or expressions are unequal; as $a^2 + b^2 > a^2$.

The members or sides of an inequality are the numbers or expressions which are connected by one of the signs of inequality, $>$ or $<$.

3. Two inequalities are of the **Same** or **Opposite Species,** or are said to subsist in the *same* or *opposite sense*, according as they have the *same* or *opposite* sign of inequality.

E.g., $8 > 3$ and $-5 > -7$ are inequalities of the same species; $0 > -1$ and $0 < 1$ are inequalities of opposite species.

4. Observe that a relation of inequality between two numbers can be stated in two ways; as $7 > 3$, or $3 < 7$.

That is, *if the members of an inequality be interchanged, the sign of inequality must be reversed.*

Principles of Inequalities.

5. *If one number be greater than a second, and this second number be greater than a third, then the first number is greater than the third;* that is,

If $a > b$ and $b > c$, then $a > c$.

In like manner, if $a < b$ and $b < c$, then $a < c$.

E.g., $3 > 2$, $2 > 1$, and $3 > 1$; $-3 < -2$, $-2 < 0$, and $-3 < 0$.

6. Addition and Subtraction. — The following principles of inequalities involve the operations of addition and subtraction:

(i.) *If the same number, or equal numbers, be added to or subtracted from both members of an inequality, the resulting inequality will be of the same species;* that is,

If $a > b$, then $a \pm m > b \pm m$.

E.g., $3 > 2$, and $3 + 1 > 2 + 1$, and $3 - 1 > 2 - 1$.

(ii.) *If the corresponding members of two or more inequalities of the same species be added, the resulting inequality will be of the same species;* that is,

If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, $\cdots$, then $a_1 + a_2 + a_3 + \cdots > b_1 + b_2 + b_3 + \cdots$.

E.g., $-5 > -7$, $3 > 2$, $0 > -4$, and $-5+3+0 > -7+2-4$; *i.e.*, $-2 > -9$.

(iii.) *If the members of one inequality be subtracted from the corresponding members of another inequality of the same species, the resulting inequality will not necessarily be of the same species;* that is,

If $a_1 > b_1$ and $a_2 > b_2$, then $a_1 - a_2$ may or may not $> b_1 - b_2$.

E.g., $11 > 6$, $4 > 3$, and $11 - 4 > 6 - 3$; $5 > 4$, $3 > 1$, but $5 - 3 < 4 - 1$.

(iv.) *If the members of an inequality be subtracted from the corresponding members of an equality, the resulting inequality will be of the opposite species;* that is, if

$$a = b, \text{ and } c > d, \text{ then } a - c < b - d.$$

E.g., $4 = 4$, $3 > -2$, and $4 - 3 < 4 - (-2)$, or $1 < 6$.

The proof of the principle enunciated in (i.) follows; the other principles are easily proved in a similar manner.

(i.) If $a > b$, then $a - b$ is positive; and $a - b \pm m \mp m$ is positive. Therefore $(a \pm m) - (b \pm m)$ is positive; and hence $a \pm m > b \pm m$.

7. Multiplication and Division. — The following principles of inequalities involve the operations of multiplication and division:

(i.) *If both members of an inequality be multiplied or divided by the same positive number, or by equal positive numbers, the resulting inequality will be of the same species;* that is, if

$$a > b, \text{ then } an > bn, \text{ and } \frac{a}{n} > \frac{b}{n},$$

wherein n is a positive number.

E.g., $-3 > -5$, and $-15 > -25$, and $-1 > -\frac{5}{3}$.

(ii.) *If both members of an inequality be multiplied or divided by the same negative number, or by equal negative numbers, the resulting inequality will be of the opposite species;* that is, if

$$a > b, \text{ then } a(-n) < b(-n), \text{ and } \frac{a}{-n} < \frac{b}{-n},$$

wherein $-n$ is a negative number.

E.g., $2 > -1$, and $2(-3) < (-1)(-3)$, or $-6 < 3$;

$$\text{and } \frac{2}{-2} < \frac{-1}{-2}, \text{ or } -1 < \frac{1}{2}.$$

(iii.) *If all the members of two or more inequalities of the same species be positive, and if the corresponding members be multiplied together, the resulting inequality will be of the same species;* that is, if

$$a_1 > b_1,\ a_2 > b_2,\ a_3 > b_3, \text{ then } a_1a_2a_3 > b_1b_2b_3,$$

wherein a_1, b_1, a_2, b_2, a_3, b_3 are all positive.

E.g., $12 > 4$, $3 > 2$, and $12 \times 3 > 4 \times 2$, or $36 > 8$.

The proof of the principle enunciated in (ii.) follows; the other principles can be easily proved in a similar way.

(ii.) If $a > b$, then $a - b$ is positive. Let $-m$ be any negative number. Then

$$-m(a-b),\ = -ma - (-mb), \text{ and } \frac{(a-b)}{-m},\ = \frac{a}{-m} - \frac{b}{-m}$$

are negative. Therefore

$$-ma < -mb, \text{ and } \frac{a}{-m} < \frac{b}{-m}.$$

8. Powers and Roots. — The following principles follow directly from those of the preceding article:

(i.) *If both members of an inequality be positive, and be raised to the same positive integral power, the resulting inequality will be of the same species;* that is, if

$$a > b, \text{ then } a^n > b^n,$$

wherein a and b are positive, and n is a positive integer.

E.g., $9 > 4$, and $81 > 16$.

(ii.) *If the same principal root of both members of an inequality be taken, the resulting inequality will be of the same species;* that is, if

$$a > b,\ \sqrt[n]{a} > \sqrt[n]{b}.$$

E.g., $9 > 4$, and $3 > 2$; $-27 < -8$, and $-3 < -2$.

9. Transformation of Inequalities. — The preceding principles enable us to make the following transformations of inequalities:

(i.) *Any term may be transferred from one member of an inequality to the other, if its sign be reversed.*

E.g., if $a - b > c$, then $a > b + c$.

(ii.) *If the signs of both members of an inequality be reversed from + to −, or from − to +, the sign of inequality must be reversed.*

E.g., $-3 < 5$, and $3 > -5$.

(iii.) *An inequality may be cleared of fractions by multiplying both members by the L. C. D., taken positively.*

E.g., if $\frac{a}{-3}-\frac{b}{5}<\frac{c}{6}$ then $-10a-6b<5c$.

If
$$\frac{x}{b-c}-\frac{y}{b+c}>\frac{z}{b^2-c^2},$$
then $x(b+c)-y(b-c)>z$, if b^2-c^2 be positive, *i.e.*, if $b>c$,
while $x(b+c)-y(b-c)<z$, if b^2-c^2 be negative, *i.e.*, if $b<c$.

(iv.) *Common positive factors can be canceled from both members of an inequality.*

E.g., $8>-12$, and $2>-3$.

If $x(a^2-b^2)<(a+b)^2$,
then $x(a-b)<(a+b)$, when $a+b$ is positive;
but $x(a-b)>(a+b)$, when $a+b$ is negative.

(v.) *If the reciprocals of the members of an inequality, which are either both positive or both negative, be taken, the resulting inequality will be of the opposite species.*

E.g., $3>2$, and $\frac{1}{3}<\frac{1}{2}$; $-5<-2$, and $-\frac{1}{5}>-\frac{1}{2}$.

10. An **Absolute Inequality** is one which holds for all values of the literal numbers involved; as $a^2+b^2>a^2$.

Such inequalities are analogous to identical equations.

A **Conditional Inequality** is one which holds only for values of the literal numbers lying between certain limits.

E.g., $x^2+1>2$, only for values of x greater than 1 and less than -1; that is, for values of x between 1 and $+\infty$, and between -1 and $-\infty$.

Absolute Inequalities.

11. Ex. 1. Prove that if $a \neq b$, then $a^2+b^2>2ab$.

We have $$(a-b)^2>0, \qquad (1)$$
since the square of any positive or negative number is positive, and therefore greater than 0.

From (1), $a^2-2ab+b^2>0$;
whence $a^2+b^2>2ab$, by Art. 9 (i.).

Ex. 2. Which is greater, $\frac{a+4b}{a+5b}$ or $\frac{a+2b}{a+3b}$, in which a and b are *positive?*

We can determine which fraction is greater by finding their difference

$$\frac{a+4b}{a+5b}-\frac{a+2b}{a+3b}=\frac{2b^2}{(a+5b)(a+3b)}.$$

Since this remainder is *positive*, we have

$$\frac{a+4b}{a+5b}>\frac{a+2b}{a+3b}.$$

Conditional Inequalities.

12. Ex. 1. Between what limits must x lie to satisfy the inequality

$$x>5x-10\,?$$

Transferring terms, $-4x>-10$; whence $x<\frac{5}{2}$, by Art. 7 (ii.).

That is, the inequality is satisfied by all values of x between $\frac{5}{2}$ and $-\infty$.

Ex. 2. What values of x satisfy the inequality

$$x^2+5x>-6\,?$$

Transferring -6, $x^2+5x+6>0$; or $(x+2)(x+3)>0$.

In order that the product $(x+2)(x+3)$ may be greater than 0, *i.e.*, *positive*, the two factors must be either *both positive* or *both negative*.

The factors $x+2$ and $x+3$ will be both positive, when $x>-2$.

Thus, if $x=-1$, then $(x+2)(x+3)=(-1+2)(-1+3)=2$.

The factors will be both negative, when $x<-3$.

Thus, if $x=-4$, then $(x+2)(x+3)=(-4+2)(-4+3)=2$.

Therefore, the given inequality will be satisfied by all values of x between -2 and $+\infty$, and between -3 and $-\infty$.

Ex. 3. What values of x and y satisfy the inequality

$$5x+3y>11, \qquad (1)$$

and the equality $3x+5y=13\,?$ (2)

Multiplying (1) by 3, $15x+9y>33$. (3)

Multiplying (2) by 5, $15x+25y=65$. (4)

Subtracting (4) from (3), $-16y>-32$, or $y<2$.

Multiplying (1) by 5, $25x+15y>55$. (5)

Multiplying (2) by 3, $9x+15y=39$. (6)

Subtracting (6) from (5), $16x>16$, or $x>1$.

Notice that not *any* value of x greater than 1 taken with *any* value of y less than 2, will satisfy both (1) and (2). But such values of x and y as satisfy (1) and (2) simultaneously, must be greater than 1 for x, and less than 2 for y. If we assign to x any value greater than 1, we can determine from (2) the corresponding value of y, which will always be less than 2; these corresponding values of x and y will then satisfy (1).

E.g., let $x = \frac{5}{3}$; then from (2), $y = \frac{8}{5}, < 2$; these values of x and y satisfy (1).

EXERCISES I.

Prove the following inequalities, in which the literal numbers are all positive and unequal:

1. $a^2 + b^2 + c^2 > ab + ac + bc$. **2.** $a^2b^2 + b^2c^2 + a^2c^2 > abc(a + b + c)$.

3. $ab(a + b) + bc(b + c) + ac(a + c) > 6\,abc$.

4. If $l^2 + m^2 + n^2 = 1$, and $l_1^2 + m_1^2 + n_1^2 = 1$, then $ll_1 + mm_1 + nn_1 < 1$.

5. $a^3 + b^3 > a^2b + ab^2$. **6.** $a^4 + b^4 > a^3b + ab^3$.

7. $(a + b)(b + c)(c + a) > 8\,abc$. **8.** $3(a^2 + b^2 + c^2) > (a + b + c)^2$.

9. $a^3 - b^3 > 3\,a^2b - 3\,ab^2$, if $a > b$; $< 3\,a^2b - 3\,ab^2$, if $a < b$.

10. $(ab + xy)(ax + by) > 4\,abxy$. **11.** $a^3 + b^3 + c^3 > 3\,abc$.

12. $a^4 + b^4 + c^4 > abc(a + b + c)$. **13.** $(a + b + c)^3 > 3(a + b)(a + c)(b + c)$.

If x be positive, which fraction is the greater:

14. $\dfrac{x + 4}{x + 3}$ or $\dfrac{x + 2}{x + 1}$? **15.** $\dfrac{x + 5}{x - 6}$ or $\dfrac{x + 3}{x - 4}$, if $x > 6$?

Determine the limits between which the values of x must lie to satisfy each of the following inequalities:

16. $x - 8 > 4$. **17.** $-3(x + 10) > -20$.

18. $\dfrac{3x - 8}{4} - x < \dfrac{37 - 2x}{3} + 9$. **19.** $\dfrac{11a - x}{4a + b} > \dfrac{a - x}{b - a}$.

20. $x - \dfrac{a}{1 - a} < 1 - \dfrac{x - 1}{a - 1}$. **21.** $\dfrac{x}{a + b} + \dfrac{x}{a - b} < 2a$.

22. $x^2 - 3x + 2 > 0$. **23.** $x^2 - x - 6 > 0$.

24. $\dfrac{x + 1}{x - 2} > 0$. **25.** $\dfrac{6x^2 - 7x + 2}{2x^2 - 5x - 3} < 0$.

Determine the limits between which the values of x must lie to satisfy simultaneously each of the following systems of inequalities:

26. $\begin{cases} 6x + 1 > 0, \\ 25 - 4x > 0. \end{cases}$ **27.** $\begin{cases} \frac{1}{8}x - \frac{1}{4}x + \frac{1}{2}x > x + 5, \\ \frac{1}{8}(x + 2) > -\frac{1}{7}(x - 2). \end{cases}$

28. $\begin{cases} x^2 - 12x + 32 > 0, \\ x^2 - 13x + 22 > 0. \end{cases}$ **29.** $\begin{cases} x^2 - 3x - 4 > 0, \\ x^2 - x - 6 > 0. \end{cases}$

What value of x satisfies each of the following systems:

30. $\begin{cases} 2x^2 - 5x + 2 = 0, \\ x^2 - 1 > 0? \end{cases}$ **31.** $\begin{cases} x^2 + x - 6 = 0, \\ x^2 + 3x - 4 > 0? \end{cases}$

Determine the limits between which the values of x and y must lie to satisfy the following systems:

32. $\begin{cases} 2x + 3y = -4, \\ x - y > 2. \end{cases}$ **33.** $\begin{cases} 7x + y = 15, \\ 3x - 2y > 14. \end{cases}$

Determine the limits between which the values of a must lie to make each of the following values of x positive:

34. $x = \dfrac{a}{11 - 2a}$. **35.** $x = \dfrac{8 - 3a}{15 - 4a}$. **36.** $x = \dfrac{4a - 1}{9 - 2a}$.

Problems.

13. Pr. 1. Divide 80 into two parts, such that the greater part shall exceed twice the sum of 4 and the less part.

Let x stand for the greater part; then $80 - x$ will stand for the less.

By the given condition,

$$x > 2(80 - x + 4), \text{ or } 3x > 168; \text{ whence } x > 56.$$

Therefore any number greater than 56 (and less than 80) will satisfy the condition of the problem.

E.g., if $x = 60$, the greater part, then $80 - x = 20$, the less part; and $60 > 2 \times 24$.

Pr. 2. A man receives from an investment an integral number of dollars a day. He calculates that if he were to receive $6 more a day his investment would yield over $270 a week; but that, if he were to receive $14 less a day, his investment would not yield as much as $270 in two weeks. How much does he receive a day from his investment?

Let x stand for the number of dollars which he receives a day.

Then, by the first condition,

$$7(x + 6) > 270; \text{ whence } x > 32\tfrac{4}{7}.$$

And, by the second condition,

$$14(x - 14) < 270; \text{ whence } x < 33\tfrac{2}{7}.$$

Therefore he receives $33 a day from his investment.

EXERCISES II.

1. What integers have each the property that one-half of the integer, increased by 5, is greater than four-thirds of it, diminished by 3?

2. What integers have each the property that, if 9 be subtracted from three times the integer, the remainder will be less than twice the integer, increased by 12?

3. A has three times as much money as B. If B gives A $10, then A will have more than seven times as much as B will have left. What are the possible amounts of money which A and B have?

4. Find a multiple of 25, such that three-fourths of it is greater than one-half of it, increased by 15, while five times the number is less than three times the number, increased by 200.

5. What positive numbers have each the property that, if the number be subtracted from a and be added to b, the product of the resulting numbers will be greater than the product of the given numbers?

6. In a class-room can be placed 6 benches, but it contains fewer. If 5 pupils be seated on each bench, then 4 pupils will be without seats. But if 6 pupils be seated on each bench, some seats will be unoccupied. How many benches are in the room?

A Property of Fractions.

14. *If the denominators of the fractions* $\frac{n_1}{d_1}, \frac{n_2}{d_2}, \frac{n_3}{d_3} \cdots$ *be all positive, then the fraction* $\frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots}$ *is greater than the least, and less than the greatest, of the given fractions.*

Let $\frac{n_1}{d_1}$ be the greatest of the given fractions, and let

$$\frac{n_1}{d_1} = x, \text{ or } n_1 = d_1x. \tag{1}$$

Then $\frac{n_2}{d_2} < x$, or $n_2 < d_2x$; $\frac{n_3}{d_3} < x$, or $n_3 < d_3x$; etc. (2)

From the equation (1), and the inequalities (2), we have

$$n_1 + n_2 + n_3 + \cdots < (d_1 + d_2 + d_3 + \cdots)x.$$

Therefore $$\frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots} < x, \text{ i.e.}, < \frac{n_1}{d_1}.$$

In like manner it can be proved that $\frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots}$ is greater than the least of the given fractions.

A Property of Powers.

15. If d be a positive number and n a positive integer, then

$$(1 + d)^n > 1 + nd.$$

We have $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$,

or $$a^n = b^n + (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Let $$a > b.$$

Then $a^{n-1} > b^{n-1}$, $a^{n-2}b > b^{n-1}$, $\cdots$, $ab^{n-2} > b^{n-1}$, $b^{n-1} = b^{n-1}$.

Therefore $a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1} > b^{n-1} + b^{n-1} + \cdots n$ terms
$> nb^{n-1}$.

Consequently, since $a - b$ is positive, $a^n > b^n + n(a - b)b^{n-1}$.

Now let $a = 1 + d$, and $b = 1$. Then $(1 + d)^n > 1 + nd$.

CHAPTER XVIII.

IRRATIONAL NUMBERS.

1. If a be the qth power of a number, say b, then $\sqrt[q]{a}, = \sqrt[q]{b^q}$, has, as we have seen in Ch. XVI., a definite value; as $\sqrt[4]{16} = 2$.

We shall now consider roots of positive numbers which are not powers with exponents equal to or multiples of the indices of the required roots.

2. *The qth root of a positive fraction, whose terms (either or both) are not qth powers of positive integers, cannot be expressed either as an integer or as a fraction.*

The proof of the general case will be first illustrated by the $\sqrt{2}$.

The $\sqrt{2}$ must be a number whose square is 2. But since $1^2 = 1$ and $2^2 = 4$, the $\sqrt{2}$ cannot be an integer.

Let us assume that $\sqrt{2}$ can be expressed as a fraction, $\frac{n}{d}$, reduced to its lowest terms. Then from

$$\sqrt{2} = \frac{n}{d}, \quad (1) \qquad \text{we have} \qquad 2 = \frac{n^2}{d^2}. \quad (2)$$

Since $\frac{n}{d}$ is in its lowest terms, $\frac{n^2}{d^2}$ is in its lowest terms [Ch. IX., Art. 29 (iii.)]. Consequently, by Ch. IX., Art. 29 (v.),

$$n^2 = 2, \text{ and } d^2 = 1. \qquad (3)$$

But since 2 is not the square of an integer, the first of equations (3), and therefore also (1), is untenable.

Consequently, $\sqrt{2}$ cannot be expressed as a fraction.

In general, $\sqrt[q]{\frac{N}{D}}$, wherein N and D (either or both) are not qth powers of positive integers, cannot be expressed as a fraction $\frac{n}{d}$.

The proof is identical with that for the $\sqrt{2}$.

Observe that, if d be assumed equal to 1, the preceding proof shows that $\sqrt[q]{\frac{N}{D}}$ cannot be expressed as a positive integer; also, if D be assumed equal to 1, that the $\sqrt[q]{N}$ cannot be expressed as a positive integer, or as a positive fraction.

3. It is therefore necessary to exclude such roots from our consideraion or to enlarge our idea of number. The latter alternative is in ccordance with the generalizing spirit of Algebra.

We therefore assume that $\sqrt{2}$, and in general, $\sqrt[q]{\frac{N}{D}}$, is a number, and nclude it in our number system.

The properties of these new numbers must be consistent with the defiition of a root; that is, with the relations,

$$(\sqrt{2})^2 = 2, \text{ and } \left(\sqrt[q]{\frac{N}{D}}\right)^q = \frac{N}{D}.$$

4. Before operating with or upon the numbers thus introduced into the number system, we must prove that they obey the fundamental laws of Algebra, which were proved in Chs. II. and III. only for integers and fractions. The following property will lead to another definition of the $\sqrt{2}$, and in general of the $\sqrt[q]{\frac{N}{D}}$, which is consistent with that given in Art. 3, and from which the fundamental laws can be easily deduced.

If $\frac{N}{D}$ be a fraction whose terms (either or both) are not qth powers of positive integers, numbers can always be found, both greater and less than $\sqrt[q]{\frac{N}{D}}$, which differ from $\sqrt[q]{\frac{N}{D}}$ by as little as we please; that is, by less than any assigned number, however small.

The proof of the general case will first be illustrated by the $\sqrt{2}$.

Since 2 lies between 1^2 and 2^2, the $\sqrt{2}$ lies between 1 and 2, *i.e.*, $1 < \sqrt{2} < 2$.

The interval between 1 and 2 we now divide into ten equal parts, and form the series of powers

$$1^2,\ 1.1^2,\ 1.2^2,\ 1.3^2,\ 1.4^2,\ 1.5^2,\ \cdots,\ 1.9^2,\ 2^2.$$

Then 2, which lies between 1^2 and 2^2, must lie between two consecutive powers of this series, or between two consecutive numbers of the equivalent series

$$1,\ 1.21,\ 1.44,\ 1.69,\ 1.96,\ 2.25,\ \cdots,\ 3.61,\ 4.$$

Since 2 lies between 1.96 and 2.25 (that is, between 1.4^2 and 1.5^2), the $\sqrt{2}$ must lie between 1.4 and 1.5, *i.e.*, $1.4 < \sqrt{2} < 1.5$.

The interval between 1.4 and 1.5, $= .1$, we next divide into ten equal parts, and form the series of powers

$$1.4^2,\ 1.41^2,\ 1.42^2,\ \cdots,\ 1.49^2,\ 1.5^2.$$

Then 2, which lies between 1.4^2 and 1.5^2, must lie between two consecutive powers of this series, or between two consecutive numbers of the equivalent series

$$1.9600,\ 1.9881,\ 2.0164,\ \cdots,\ 2.2201,\ 2.2500.$$

Since 2 lies between 1.9881 and 2.0164 (that is, between 1.41^2 and 1.42^2), the $\sqrt{2}$ lies between 1.41 and 1.42, *i.e.*, $1.41 < \sqrt{2} < 1.42$.

This method of procedure can be continued indefinitely. These results may be summarized as follows:

(*a*)		(*b*)
$1 < \sqrt{2} < 2$	and	$2 - 1 = 1$
$1.4 < \sqrt{2} < 1.5$	and	$1.5 - 1.4 = .1$
$1.41 < \sqrt{2} < 1.42$	and	$1.42 - 1.41 = .01$
$1.414 < \sqrt{2} < 1.415$	and	$1.415 - 1.414 = .001$, etc.

It follows from tables (*a*) and (*b*) that there can be found two numbers, one greater and the other less than $\sqrt{2}$, which differ from each other by as little as we please, and which therefore differ from $\sqrt{2}$, which lies between them, by as little as we please.

Observe that the numbers of the one series, which are always less than $\sqrt{2}$, continually *increase* toward $\sqrt{2}$, while the numbers of the other series, which are always greater than $\sqrt{2}$, continually *decrease* toward $\sqrt{2}$.

Either of these two values is an approximation to $\sqrt{2}$.

In the proof of the general case, which now follows, it is necessary to represent the two values between which the required root lies at any stage of the work in terms of common fractions instead of decimal fractions.

Thus, $$1.414 < \sqrt{2} < 1.415$$
could have been written
$$\frac{14}{10} + \frac{1}{10^2} + \frac{4}{10^3} < \sqrt{2} < \frac{14}{10} + \frac{1}{10^2} + \frac{5}{10^3}.$$

Let $\frac{N}{D}$ be a fraction, in which N and D (either or both) are not qth powers of positive integers. Evidently the powers
$$0^q,\ \left(\frac{1}{10}\right)^q,\ \left(\frac{2}{10}\right)^q,\ \left(\frac{3}{10}\right)^q,\ \cdots,\ \left(\frac{n}{10}\right)^q,\ \cdots$$
increase without limit. Therefore, whatever positive value $\frac{N}{D}$ may have, there will always be two consecutive powers of the above series between which $\frac{N}{D}$ lies. Let $\left(\frac{k_1}{10}\right)^q$ and $\left(\frac{k_1+1}{10}\right)^q$ be the two powers between which $\frac{N}{D}$ is found to lie, wherein k_1 is 0 or any *positive integer.*

Then since $\frac{N}{D}$ lies between $\left(\frac{k_1}{10}\right)^q$ and $\left(\frac{k_1+1}{10}\right)^q$, the $\sqrt[q]{\frac{N}{D}}$ lies between $\frac{k_1}{10}$ and $\frac{k_1+1}{10}$; *i.e.*,
$$\frac{k_1}{10} < \sqrt[q]{\frac{N}{D}} < \frac{k_1+1}{10}. \qquad \text{(I.)}$$

The interval between $\frac{k_1+1}{10}$ and $\frac{k_1}{10}$, $=\frac{1}{10}$, we now divide into ten equal parts, and form the series of powers

$$\left(\frac{k_1}{10}\right)^q,\ \left(\frac{k_1}{10}+\frac{1}{10^2}\right)^q,\ \left(\frac{k_1}{10}+\frac{2}{10^2}\right)^q,\ \cdots,\ \left(\frac{k_1}{10}+\frac{9}{10^2}\right)^q,\ \left(\frac{k_1+1}{10}\right)^q.$$

Then $\frac{N}{D}$, which lies between $\left(\frac{k_1}{10}\right)^q$ and $\left(\frac{k_1+1}{10}\right)^q$, must lie between two consecutive powers of this series.

Let $\left(\frac{k_1}{10}+\frac{k_2}{10^2}\right)^q$ and $\left(\frac{k_1}{10}+\frac{k_2+1}{10^2}\right)^q$, wherein k_2 is one of the numbers, 0, 1, 2, $\cdots$, 8, 9, be the two powers between which $\frac{N}{D}$ is found to lie.

Then $\sqrt[q]{\frac{N}{D}}$ lies between $\frac{k_1}{10}+\frac{k_2}{10^2}$ and $\frac{k_1}{10}+\frac{k_2+1}{10^2}$; *i.e.*,

$$\frac{k_1}{10}+\frac{k_2}{10^2}<\sqrt[q]{\frac{N}{D}}<\frac{k_1}{10}+\frac{k_2+1}{10^2}. \qquad \text{(II.)}$$

The method can evidently be carried on indefinitely; that is, we can find two powers

$$\left(\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}\right)^q \text{ and } \left(\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}\right)^q$$

between which $\frac{N}{D}$ lies. We therefore have

$$\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}<\sqrt[q]{\frac{N}{D}}<\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}, \qquad \text{(III.)}$$

wherein p is any positive integer from 1 to $+\infty$.

The two numbers between which $\sqrt[q]{\frac{N}{D}}$ is found to lie at any stage of the work evidently differ by $\frac{1}{10^p}$. As p increases without limit, $\frac{1}{10^p}$ decreases without limit (Ch. III., § 4, Art. 19). Since, therefore, these two numbers can be made to differ from each other by less than any assigned number, however small, the $\sqrt[q]{\frac{N}{D}}$, which lies between them, will differ from either of them by less than any assigned number, however small.

Either of these numbers is an approximate value of $\sqrt[q]{\frac{N}{D}}$.

5. It is important to keep clearly in mind that, although approximate values have been obtained for $\sqrt{2}$, and in general for $\sqrt[q]{\frac{N}{D}}$, these numbers have as exact values as have integers and fractions.

Thus, $\sqrt{2}\times\sqrt{2}=2$, by definition of a root. Now no approximate value of $\sqrt{2}$ multiplied by itself gives exactly 2. Therefore the number which multiplied by itself gives 2 must have an exact value. This exact value, to be sure, cannot be expressed in terms of integers and fractions.

6. In Art. 4 we found that the $\sqrt[q]{\frac{N}{D}}$ lies between two corresponding numbers of the two series:

$$\frac{k_1}{10},\ \frac{k_1}{10}+\frac{k_2}{10^2},\ \cdots,\ \frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}, \tag{1}$$

$$\frac{k_1+1}{10},\ \frac{k_1}{10}+\frac{k_2+1}{10^2},\ \cdots,\ \frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}, \tag{2}$$

wherein $p = 1, 2, 3, \cdots, \infty$.

These two series have the following properties:

(i.) *The numbers*

$$\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p},\ \textit{wherein}\ p = 1, 2, 3, \cdots, \infty,$$

of the first series increase as p increases, but remain always less than the numbers

$$\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p},\ \textit{wherein}\ p = 1, 2, 3, \cdots, \infty,$$

of the second series; and the numbers of the second series decrease as p increases, but remain always greater than the numbers of the first series. That is, the numbers of the one series more and more nearly approach the numbers of the other series, but never meet them.

(ii.) *The difference between a number of the one series and the corresponding number of the other series can be made less than any assigned number, however small, by taking p sufficiently great.*

7. Two series of numbers which possess the properties (i.) and (ii.), Art. 6, are said to have a common limit, which lies between them. Two such series therefore define the number which is their common limit. This number is approached by both series and not reached by either.

The two numbers between which the $\sqrt[q]{\frac{N}{D}}$ lies can be reduced to a common denominator 10^p. Let us designate 10^p by n. Then since these two numbers differ by $\frac{1}{10^p}$, $=\frac{1}{n}$, they may be represented by $\frac{m}{n}$ and $\frac{m+1}{n}$ respectively. In the theory which follows, we shall let

$$\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}=\frac{m}{n}; \quad (1) \qquad \frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}=\frac{m+1}{n}. \quad (2)$$

That is, $$\frac{m}{n}<\sqrt[q]{\frac{N}{D}}<\frac{m+1}{n}. \tag{I.}$$

Irrational Numbers.

8. An **Irrational Number** is a number which cannot be expressed either as an integer or as a fraction, but which can be inclosed between two fractions ultimately differing from each other, and therefore from the inclosed number, by less than any assigned number however small.

An irrational number, I, is therefore defined by the relation

$$\frac{m}{n} < I < \frac{m+1}{n},$$

wherein $\frac{m}{n}$ and $\frac{m+1}{n}$ have the properties (i.) and (ii.), Art. 6; as $\sqrt{2}$

9. Whatever value p, and therefore $\frac{m}{n}$ and $\frac{m+1}{n}$, may have, there will always be numbers, integers or fractions, lying between any two numbers of the series $\frac{m}{n}$ and $\frac{m+1}{n}$. But no such number can be selected which will not be passed by numbers of one or the other series, if p be sufficiently increased. Therefore there is no number in the system defined so as to include only integers and fractions, which is greater than every number of the series (1.) and less than every number of the series (2.); that is, which is approached by both series and not reached by either. Since, however, these series cannot meet, we conclude that there was a gap between them which could not be filled by any integer or fraction. Consequently by including irrational numbers in the number system, continuity has been introduced where before it was lacking.

Negative Irrational Numbers.

10. If the fractions of the series which define an irrational number be negative, the number thus defined is called a **Negative Irrational Number.** Therefore a negative irrational number is defined by the relation

$$-\frac{m+1}{n} < -I < -\frac{m}{n},$$

wherein the two series of fractions, $-\frac{m+1}{n}$ and $-\frac{m}{n}$, have the properties (i.) and (ii.), Art. 6.

11. The positive and negative irrational numbers defined by the relations

$$\frac{m}{n} < I < \frac{m+1}{n}, \quad -\frac{m+1}{n} < -I < -\frac{m}{n}$$

are called *equal and opposite.* The absolute value of an irrational number is its value without regard to quality.

The Fundamental Operations with Irrational Numbers.

12. Addition. — Let I_1 and I_2 be two positive irrational numbers defined by the relations

$$\frac{m_1}{n_1} < I_1 < \frac{m_1+1}{n_1}, \quad (1) \qquad \frac{m_2}{n_2} < I_2 < \frac{m_2+1}{n_2}. \quad (2)$$

If the corresponding *rational* numbers of the series which define I_1 and I_2 be added, we obtain the two series of rational numbers

$$\frac{m_1}{n_1}+\frac{m_2}{n_2} \text{ and } \frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}.$$

The numbers of these series have the properties (i.) and (ii.), Art. 6. For, since $\frac{m_1}{n_1}$ increases as n_1 increases, and $\frac{m_2}{n_2}$ increases as n_2 increases, therefore $\frac{m_1}{n_1}+\frac{m_2}{n_2}$ increases as n_1 and n_2 increase. For a similar reason, $\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}$ decreases as n_1 and n_2 increase. And since

$$\frac{m_1}{n_1}<\frac{m_1+1}{n_1} \text{ and } \frac{m_2}{n_2}<\frac{m_2+1}{n_2}, \quad \frac{m_1}{n_1}+\frac{m_2}{n_2}<\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}.$$

The difference

$$\left(\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}\right)-\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}\right), \ =\frac{1}{n_1}+\frac{1}{n_2},$$

can be made less than any assigned number, however small. For $\frac{1}{n_1}$ can be made less than any assigned number, say $\frac{1}{2}d$; and $\frac{1}{n_2}$ can be made less than any assigned number, say $\frac{1}{2}d$. Therefore, $\frac{1}{n_1}+\frac{1}{n_2}$ can be made less than $\frac{1}{2}d+\frac{1}{2}d, =d$.

Therefore, the two series of numbers

$$\frac{m_1}{n_1}+\frac{m_2}{n_2} \text{ and } \frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}$$

determine a positive number which lies between them. This number is defined as the sum I_1+I_2. That is,

$$\frac{m_1}{n_1}+\frac{m_2}{n_2}<I_1+I_2<\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}.$$

Exactly similar reasoning will apply if either or both of the irrational numbers be negative, or either be rational.

13. Subtraction. — The following definition of Subtraction of irrational numbers is a natural extension of the principle of subtraction for rational numbers.

To subtract an irrational number from a rational or irrational number is equivalent to adding an equal and opposite irrational number.

The Associative and Commutative Laws for Addition and Subtraction of Irrational Numbers.

14. These fundamental laws hold also for irrational numbers; that is

$$I_1+I_2=I_2+I_1,$$

$$I_1+I_2+I_3, =I_1+(I_2+I_3)=I_1+(I_3+I_2)=\text{etc.}$$

For, by the definition of $I_1 + I_2$,

$$\frac{m_1}{n_1}+\frac{m_2}{n_2}<I_1+I_2<\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2};$$

nd, by definition of $I_2 + I_1$,

$$\frac{m_2}{n_2}+\frac{m_1}{n_1}<I_2+I_1<\frac{m_2+1}{n_2}+\frac{m_1+1}{n_1}.$$

But since

$$\frac{m_2}{n_2}+\frac{m_1}{n_1}=\frac{m_1}{n_1}+\frac{m_2}{n_2},$$

nd

$$\frac{m_2+1}{n_2}+\frac{m_1+1}{n_1}=\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2},$$

herefore $I_1 + I_2 = I_2 + I_1$.

The Associative Law can be proved in a similar manner.

E.g., $\sqrt{2}+\sqrt{3}=\sqrt{3}+\sqrt{2}$, $\sqrt{2}+\sqrt{3}+(-\sqrt{5})=\sqrt{2}+(-\sqrt{5})+\sqrt{3}$.

15. Multiplication.—Let I_1 and I_2 be two positive irrational numbers defined by the relations

$$\frac{m_1}{n_1}<I_1<\frac{m_1+1}{n_1}, \quad (1) \qquad \frac{m_2}{n_2}<I_2<\frac{m_2+1}{n_2}. \quad (2)$$

If the corresponding *rational* numbers of the series which define I_1 and I_2 be multiplied, we obtain the two series of rational numbers,

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2} \text{ and } \frac{m_1+1}{n_1}\cdot\frac{m_2+1}{n_2}.$$

The numbers of these series have the properties (i.) and (ii.), Art. 6.

For, since $\frac{m_1}{n_1}$ increases as n_1 increases, and $\frac{m_2}{n_2}$ increases as n_2 increases, therefore $\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}$ increases as n_1 and n_2 increase. For a similar reason $\frac{m_1+1}{n_1}\cdot\frac{m_2+1}{n_2}$ decreases as n_1 and n_2 increase. And since

$$\frac{m_1}{n_1}<\frac{m_1+1}{n_1} \text{ and } \frac{m_2}{n_2}<\frac{m_2+1}{n_2}, \text{ therefore } \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}<\frac{m_1+1}{n_1}\cdot\frac{m_2+1}{n_2}.$$

The difference

$$\frac{m_1+1}{n_1}\cdot\frac{m_2+1}{n_2}-\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}, =\frac{m_1+1}{n_1}\cdot\frac{m_2+1}{n_2}-\frac{m_1+1}{n_1}\cdot\frac{m_2}{n_2}+\frac{m_1+1}{n_1}\cdot\frac{m_2}{n_2}$$

$$-\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{m_1+1}{n_1}\left(\frac{m_2+1}{n_2}-\frac{m_2}{n_2}\right)+\frac{m_2}{n_2}\left(\frac{m_1+1}{n_1}-\frac{m_1}{n_1}\right),$$

can be made less than any assigned number, however small.

For, since $\frac{m_1+1}{n_1}$ decreases, it is always less than some positive finite rational number, say R; and since $\frac{m_2}{n_2}<\frac{m_2+1}{n_2}$, it is also less than some positive finite rational number, say R_1. Moreover, $\frac{m_2+1}{n_2}-\frac{m_2}{n_2}$ and $\frac{m_1+1}{n_1}-\frac{m_1}{n_1}$ can each be made less than any assigned number, say d.

Therefore the given difference can be made less than $dR + dR_1$, $= d(R + R_1)$. But $d(R + R_1)$ can be made less than any assigned number, say δ, by taking d less than $\frac{\delta}{R + R_1}$.

Therefore the two series $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2}$ and $\frac{m_1 + 1}{n_1} \cdot \frac{m_2 + 1}{n_2}$, determine a positive number which lies between them. This number is defined as the product $I_1 \cdot I_2$. That is,

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} < I_1 \cdot I_2 < \frac{m_1 + 1}{n_1} \cdot \frac{m_2 + 1}{n_2}.$$

The following definition of multiplication of irrational numbers is consistent with the preceding result.

The product of two irrational numbers is the product of their absolute values, with a sign determined by the laws of signs for the product of two rational numbers.

The Associative, Commutative, and Distributive Laws for Multiplication of Irrational Numbers.

16. These fundamental laws hold also for irrational numbers. That is,

$$I_1 I_2 = I_2 I_1\,; \quad I_1 I_2 I_3 = I_1(I_2 I_3) = \text{etc.}\,; \quad (I_1 \pm I_2) I_3 = I_1 I_3 \pm I_2 I_3.$$

The proofs of these principles are similar to those given in Art. 14.

17. Reciprocal of an Irrational Number. — It can easily be proved that the reciprocal of the numbers of the series which define I have the properties (i.) and (ii.), Art. 6.

Therefore the two series of numbers, $\frac{1}{\frac{m + 1}{n}}$ and $\frac{1}{\frac{m}{n}}$, define a positive number which lies between them. This number is defined as the reciprocal of I. That is, $\frac{1}{\frac{m + 1}{n}} < \frac{1}{I} < \frac{1}{\frac{m}{n}}$.

18. Division. — Division by an irrational number may be defined as follows:

To divide any number by an irrational number, not 0, is equivalent to multiplying it by the reciprocal of the irrational number.

From this definition it follows that the fundamental laws hold also for division of irrational numbers.

19. It follows from the preceding theory that the laws governing the fundamental operations with irrational numbers are the same as those governing these operations with rational numbers.

E.g., $\sqrt{2} - (\sqrt{3} - \sqrt{5}) = \sqrt{2} - \sqrt{3} + \sqrt{5}$; $(\sqrt{2}\sqrt{3})^3 = (\sqrt{2})^3(\sqrt{3})^3$.

CHAPTER XIX

SURDS.

1. In Ch. XVI. we considered only roots whose radicands are powers with exponents equal to or multiples of the indices of the roots.

In Ch. XVIII. we assumed the existence of roots of numbers which are not powers with exponents equal to or multiples of the indices of the required roots, and proved that such roots obey the fundamental laws of Algebra; as $\sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$, etc.

Such roots were called **Irrational Numbers.**

2. A **Rational Number** is a number which can be expressed as an integer or as a fraction; as $2, \frac{2x}{3y}, \sqrt[3]{(27a^6)}$.

A **Rational Expression** is an expression which involves only rational numbers; as $\frac{2}{3}a + \frac{1}{2}b, ab + \sqrt{a^2}$.

3. A **Radical** is an indicated root of a number or expression; as $\sqrt{7}, \sqrt{9}, \sqrt[3]{(a+b)}$.

A **Radical Expression** is an expression which contains radicals; as $2\sqrt{7}, \sqrt{x} + \sqrt{y}, \sqrt{(a+\sqrt{b})}$.

A **Surd** is an irrational root of a rational number; as $\sqrt{7}, \sqrt{a}$.

Observe that $\sqrt{(1+\sqrt{7})}$ is not a surd, since $1+\sqrt{7}$ is not a rational number.

Notice the difference between arithmetical and algebraical irrationality. Thus, $\sqrt{a}$ is algebraically irrational; but if $a = 4$, then $\sqrt{a}, = \sqrt{4}, = 2$, is arithmetically rational.

Classification of Surds.

4. A **Quadratic Surd,** or a **Surd of the Second Order,** is one with index 2; as $\sqrt{3}, \sqrt{a}$.

A **Cubic Surd, or a Surd of the Third Order,** is one with index 3; as $\sqrt[3]{(a+b)}$, $\sqrt[3]{7}$.

A **Biquadratic Surd, or a Surd of the Fourth Order,** is one with index 4; as $\sqrt[4]{(ab)}$, $\sqrt[4]{5}$.

A **Simple Monomial Surd Number** is a single surd number, or a rational multiple of a single surd number; as $\sqrt{3}$, $2\sqrt{5}$.

A **Simple Binomial Surd Number** is the sum of two simple surd numbers, or of a rational number and a simple surd number; as $\sqrt{2}+\sqrt[3]{3}$, $3+\sqrt{6}$.

5. The principles enunciated in Ch. XVI., and their proofs, hold also for irrational roots. Each principle will be restated as occasion for its use arises in this chapter. As in Ch. XVI., we shall limit the radicands to positive values, and the roots to principal roots.

Reduction of Surds.

6. A surd is in its *simplest form* when the radicand is integral, and does not contain a factor with an exponent equal to or a multiple of the index of the root; as $\sqrt{2}$, $\sqrt[3]{(a^2b)}$, $\sqrt[n]{a^m}$.

A surd can be reduced to its simplest form by applying one or more of the following principles:

(i.) $\sqrt[q]{a^{kq}} = a^{\frac{kq}{q}} = a^k$ [Ch. XVI., § 1, Art. 13 (ii.)]

(ii.) $\sqrt[q]{(ab)} = \sqrt[q]{a} \times \sqrt[q]{b}$ [Ch. XVI., § 1, Art. 13 (iii.)]

(iii.) $\sqrt[q]{\frac{a}{b}} = \frac{\sqrt[q]{a}}{\sqrt[q]{b}}$ [Ch. XVI., § 1, Art. 13 (iv.)]

(iv.) *In a root of a power (or a power of a root) the index of the root and the exponent of the power may both be multiplied or divided by one and the same number;* or, stated symbolically,

$$\sqrt[q]{a^p} = \sqrt[kq]{a^{kp}}, \text{ and } \sqrt[q]{a^p} = \sqrt[\frac{q}{k}]{a^{\frac{p}{k}}}.$$

E.g., $\sqrt[3]{a^2} = \sqrt[6]{a^4}$; $\sqrt[6]{a^9} = \sqrt{a^3}$.

The proof is left as an exercise for the student.

7. The following examples will illustrate the methods of reducing surds to their simplest forms:

Ex. 1. $\sqrt{(18a^5b^2)} = \sqrt{(9a^4b^2)} \times \sqrt{(2a)} = 3a^2b\sqrt{(2a)}$.

Ex. 2. $\sqrt[n]{(a^{n+1}b^{2n+2})} = \sqrt[n]{(a^nb^{2n})} \times \sqrt[n]{(ab^2)} = ab^2\sqrt[n]{(ab^2)}$.

Observe that the radicand is separated into two factors, one f which is a power with the highest exponent which is equal o or a multiple of the index of the required root. The result s then obtained by multiplying the rational root of this factor y the irrational root of the second factor.

Ex. 3. $\sqrt{\frac{3a^2}{4b^2}} = \frac{\sqrt{(3a^2)}}{\sqrt{(4b^2)}} = \frac{\sqrt{a^2} \times \sqrt{3}}{\sqrt{(4b^2)}} = \frac{a\sqrt{3}}{2b}$.

When the required root of the denominator of a fraction annot be expressed rationally, multiply both terms of the raction by the expression of lowest degree which will make he denominator a power with an exponent equal to the index of the root.

Ex. 4. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$.

Ex. 5. $\sqrt[n]{\frac{a}{b^5}} = \sqrt[n]{\frac{ab^{n-5}}{b^n}} = \frac{\sqrt[n]{(ab^{n-5})}}{b}$.

By Art. 6 (iv.), a given surd can frequently be reduced to an equivalent surd of a lower order.

Ex. 6. $\sqrt[6]{(27a^3b^6)} = \sqrt[6]{b^6} \times \sqrt[6]{(3a)^3} = b\sqrt{(3a)}$.

EXERCISES I.

Reduce each of the following surds to its simplest form:

1. $\sqrt{32}$. **2.** $\sqrt{75}$. **3.** $\sqrt{108}$. **4.** $\sqrt{x^3}$.

5. $\sqrt{(a^2b)}$. **6.** $\sqrt{(a^4b^5)}$. **7.** $\sqrt{(4a^7x^{11})}$. **8.** $\sqrt{(50ax^2y^3)}$.

9. $\sqrt{(a^2b^2 + a^2c^2)}$. **10.** $\sqrt{(ab^3c^4 - b^2c^6)}$.

11. $\sqrt{(b-c)(b^3-c^3)}$. **12.** $\sqrt{(a^2-1)(1+a)}$.

13. $\sqrt{(9x^3 - 18x^2 + 9x)}$. **14.** $\sqrt{(4a^3b - 8a^2b^2 + 4ab^3)}$.

15. $\sqrt[3]{192}$. **16.** $\sqrt[3]{(-10\frac{1}{8})}$. **17.** $\sqrt[3]{(-a^{10})}$. **18.** $\sqrt[3]{(a^6b^3c^4)}$.

19. $\sqrt[3]{(16a^5x^9)}$. **20.** $\sqrt[3]{(32a^{m+6n})}$. **21.** $\sqrt[3]{(-54x^{4n+11}y^{14})}$.

22. $\sqrt[3]{(-a^{7x}b^{3x})}$. **23.** $\sqrt[3]{(a^6 - a^3x^2)}$. **24.** $\sqrt[n]{(a^{6n}b^n - a^{7n})}$.

25. $\sqrt[5]{729}$. **26.** $\sqrt[4]{(4a^6x^8)}$. **27.** $\sqrt[4]{(a^{x+11}b^{4x+1})}$. **28.** $\sqrt[6]{(a^{18}x^7)}$.

29. $\sqrt[9]{(-a^{15n}b^{10n})}$. **30.** $\sqrt[n]{(a^{2n+1}b)}$. **31.** $\sqrt[n]{(a^{nx+m}b^{3n}c^3)}$. **32.** $\sqrt[8]{[(a^2)^4 + a^2]^4}$.

33. $\sqrt{\frac{9}{10}}$. **34.** $\sqrt{\frac{8}{27}}$. **35.** $\sqrt{\frac{x^2}{a}}$. **36.** $\sqrt{\frac{3a^2}{4}}$.

37. $\sqrt{\frac{64\,a}{81\,b}}$. 38. $\sqrt{\frac{18\,a^2x^3}{125\,b^5}}$. 39. $\sqrt{\frac{4\,a^2x}{9\,b^4y^5}}$. 40. $\sqrt{\frac{16\,a^6}{45\,b^3x^5}}$.

41. $\sqrt[3]{\frac{8}{9}}$. 42. $\sqrt[3]{\frac{a}{b^3}}$. 43. $\sqrt[3]{\frac{a}{27\,b}}$. 44. $\sqrt[3]{\frac{3\,a^2x^3}{4\,b^3y^4}}$.

45. $\sqrt[3]{\frac{ax^{4n}}{8\,b^2}}$. 46. $\sqrt[3]{\frac{128\,a^7x^8}{b^6y^{18}}}$. 47. $\sqrt[3]{\frac{ax^6-a^2x^7}{64\,b^{12}}}$. 48. $\sqrt[3]{\left(1-\frac{1}{a^3}\right)}$.

49. $\sqrt{\left(\frac{a^3}{x^4}-\frac{a^5}{x^6}\right)}$. 50. $\sqrt[4]{\frac{16\,a^5x^{16}}{b^8y^{11}}}$. 51. $\sqrt[5]{\frac{a^6b^8}{x^5}}$. 52. $\sqrt[6]{\frac{a^6}{6\,b^7x^{12n}}}$.

53. $\sqrt[2n]{\frac{an^2}{x^{6n}}}$. 54. $\sqrt[4]{25}$. 55. $\sqrt[4]{(81\,a^2)}$. 56. $\sqrt[6]{a^8}$.

57. $\sqrt[14]{(4\,a^{12})}$. 58. $\sqrt[4]{\frac{9}{36}}$. 59. $\sqrt[4]{\frac{a^2}{b^2}}$. 60. $\sqrt[10]{\frac{32}{x^{15}y^{20}}}$. 61. $\sqrt[mx]{\frac{1}{a^{nx}}}$.

62. $\sqrt[6]{(8\,a^9b^{15})}$. 63. $\sqrt[12]{(64\,a^8x^{10})}$. 64. $\sqrt[15x]{(a^{30x}b^{20})}$.

65. $\sqrt[12n]{(81\,a^8b^{60})}$. 66. $\sqrt[3n]{(a^{2n+n^2}b^n)}$. 67. $\sqrt[10]{(a^2-2\,ax+x^2)^3}$.

Addition and Subtraction of Surds.

8. Similar or Like Surds are rational multiples of one and the same simple monomial surd, as $\sqrt{12}, =2\sqrt{3}$, and $5\sqrt{3}$.

Like surds, or such surds as can be reduced to *like* surds, can be united by algebraic addition into a single like surd.

Ex. 1. $\sqrt{12}+2\sqrt{27}-9\sqrt{48}=2\sqrt{3}+6\sqrt{3}-36\sqrt{3}=-28\sqrt{3}.$

Ex. 2. $$8\sqrt[3]{40}+3\sqrt[3]{135}-2\sqrt[3]{625}=16\sqrt[3]{5}+9\sqrt[3]{5}-10\sqrt[3]{5}$$
$$=15\sqrt[3]{5}.$$

Ex. 3. $\sqrt{2}-\sqrt{\frac{1}{2}}+\sqrt{.02}=\sqrt{2}-\frac{1}{2}\sqrt{2}+\frac{1}{10}\sqrt{2}=\frac{3}{5}\sqrt{2}.$

Ex. 4. $$\sqrt{(a^5b)}+2\sqrt{(a^3b^3)}+\sqrt{(ab^5)}$$
$$=a^2\sqrt{(ab)}+2\,ab\sqrt{(ab)}+b^2\sqrt{(ab)}=(a+b)^2\sqrt{(ab)}.$$

EXERCISES II.

Simplify each of the following expressions:

1. $\sqrt{24}-\sqrt{6}+\sqrt{150}.$ 2. $2\sqrt{8}+5\sqrt{72}-7\sqrt{18}.$

3. $\sqrt{54}+2\sqrt{24}-9\sqrt{96}.$ 4. $5\sqrt{3}-2\sqrt{48}+5\sqrt{108}.$

5. $3\sqrt{75}+4\frac{1}{2}\sqrt{192}-2\frac{3}{4}\sqrt{12}.$ 6. $\sqrt{2\frac{9}{20}}+\sqrt{5\frac{41}{80}}-\sqrt{\frac{9}{5}}.$

7. $4\sqrt{\frac{3}{4}}-\frac{2}{7}\sqrt{\frac{3}{16}}-2\sqrt{27}.$ 8. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}.$

9 $8\sqrt[3]{48}+3\sqrt[3]{162}-2\sqrt[3]{384}.$ 10. $5\sqrt[3]{54}+9\sqrt[3]{250}-\sqrt[3]{686}.$

11. $2\frac{3}{5}\sqrt[3]{500}+\frac{3}{4}\sqrt[3]{256}-3\frac{1}{2}\sqrt[3]{32}-\frac{2}{3}\sqrt[3]{108}.$

12. $1.5\sqrt[3]{1\frac{3}{5}}-2\frac{1}{4}\sqrt[3]{12.8}-3\frac{2}{3}\sqrt[3]{5\frac{2}{5}}+4.6\sqrt[3]{43.2}.$

13. $\sqrt[3]{40}-5\sqrt[3]{\frac{1}{25}}+4\sqrt[3]{(-.625)}+\frac{2}{3}\sqrt[3]{16\frac{7}{8}}.$

14. $2\sqrt{3}-\sqrt{12}+\sqrt[4]{9}$. **15.** $\sqrt[3]{24}+3\sqrt[6]{9}-5\sqrt[3]{192}$.

16. $\sqrt{(4a^3)}+\sqrt{(9a^3)}+\sqrt{(25a^3)}-\sqrt{(81a^3)}$.

17. $\sqrt{(12a^2b)}+\sqrt{(75a^2b)}-\sqrt{(27a^2b)}$.

18. $\sqrt[3]{(64a^3b^5)}+\sqrt[3]{(125a^3b^5)}-\sqrt[3]{(a^3b^5)}$.

19. $a\sqrt{(a^3b^7)}+b^2\sqrt{(a^5b^3)}-2ab^2\sqrt{(a^3b^3)}+\sqrt{(a^{21}b^{25})}$.

20. $3z\sqrt[3]{(250x^4z^2)}-5x\sqrt[3]{(128xz^5)}+3xz\sqrt[3]{(16xz^2)}$.

21. $3a^2b\sqrt[3]{(32a^2b)}+5\sqrt[3]{(108a^8b^4)}-ab\sqrt[3]{(500a^5b)}$.

22. $\sqrt[4]{(9a^2b^2)}+\sqrt{(27a^3b)}+5\sqrt[4]{(729a^6b^2)}$.

23. $2\sqrt[3]{(3x^2y)}-\sqrt[6]{(9x^4y^2)}+\sqrt[3]{(125x^4y)}-\sqrt[6]{(x^8y^2)}$.

24. $\sqrt{(9a+27)}+3\sqrt{(4a+12)}$.

25. $\sqrt{(4a^3+4a^2b)}+\sqrt{(4ab^2+4b^3)}$.

26. $7x\sqrt{(25a+75)}-5\sqrt{(9x^2a+27x^2)}$.

27. $2\sqrt{(2x^3)}-\sqrt{(8x)}-\sqrt{(2x^3-4x^2+2x)}$.

28. $\sqrt[5]{(a^3b^5+3b^6)}+a\sqrt[5]{(32a^3+96b)}-\sqrt[5]{(a^8+3a^5b)}$.

29. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}$.

30. $3a\sqrt{\frac{a-x}{a+x}}-3x\sqrt{\frac{a-x}{a+x}}-2a\sqrt{\frac{a-x}{a+x}}+4x\sqrt{\frac{a-x}{a+x}}$.

Reduction of Surds of Different Orders to Equivalent Surds of the Same Order.

9. Surds of different orders can be reduced to equivalent surds of the same order by the principle given in Art. 6 (iv.):

$$\sqrt[q]{a^q}=\sqrt[kq]{a^{kq}} \qquad \text{[Art. 6 (iv.)]}$$

Ex. Reduce $\sqrt{3}$, $\sqrt[4]{(2a)}$, and $\sqrt[6]{(5b)}$ to equivalent surds of the same order.

We have
$$\sqrt{3}=\sqrt[12]{3^6}=\sqrt[12]{729};$$
$$\sqrt[4]{(2a)}=\sqrt[12]{(2a)^3}=\sqrt[12]{(8a^3)};$$
$$\sqrt[6]{(5b)}=\sqrt[12]{(5b)^2}=\sqrt[12]{(25b^2)}.$$

Observe that the L. C. M. of the given indices is taken as the common index of the equivalent surds, and that each radicand is raised to a power whose exponent is equal to the quotient of this L. C. M. divided by the index of the given root.

10. Any rational number can be expressed in the form of a surd by writing under the radical sign a power of the number whose exponent is equal to the index.

E.g., $$2 = \sqrt{4} = \sqrt[3]{8} = \cdots = \sqrt[q]{2^q}.$$

11. Two surds, or a surd and a rational number, can be compared by first reducing them to equivalent surds of the same order, and then comparing the resulting radicands.

Ex. Which is greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

We have $\sqrt{2} = \sqrt[6]{8}$, and $\sqrt[3]{3} = \sqrt[6]{9}$.

Since $9 > 8$, therefore $\sqrt[6]{9} > \sqrt[6]{8}$, or $\sqrt[3]{3} > \sqrt{2}$.

EXERCISES III.

Reduce to equivalent surds of the same order:

1. $\sqrt{2}$, $\sqrt[4]{5}$. 2. $\sqrt{3}$, $\sqrt[4]{6}$. 3. $\sqrt{7}$, $\sqrt[3]{10}$.
4. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{4}}$. 5. 5, $\sqrt[4]{10}$. 6. 6, $\sqrt[3]{4}$.
7. $\sqrt[6]{2}$, $\sqrt[8]{3}$. 8. $\sqrt[10]{15}$, $\sqrt[15]{10}$. 9. $\sqrt[n]{(3a^2)}$, b^4.
10. $\sqrt[n]{a^3}$, $\sqrt[m]{b^5}$. 11. $\sqrt[n+1]{(x^3y)}$, $\sqrt[n-1]{(xy^3)}$. 12. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[4]{15}$.
13. $\sqrt[4]{2}$, 3, $\sqrt[8]{5}$. 14. $\sqrt[m]{a^2}$, b^3, $\sqrt[p]{c^4}$. 15. $\sqrt[2m]{a^3}$, $\sqrt[4m]{b^3}$, $\sqrt[6m]{c^5}$.

Which is the greater,

16. $2\sqrt{3}$ or $3\sqrt{2}$? 17. $\sqrt{5}$ or $\sqrt[3]{10}$? 18. $\frac{1}{2}\sqrt[3]{25}$ or $\frac{1}{3}\sqrt{11}$?
19. $\sqrt[3]{a^2}$ or $\sqrt{a}$, when $a < 1$? 20. $\sqrt[4]{x^3}$ or $\sqrt[5]{x^4}$, when $x > 1$?

Which is the greatest,

21. $\sqrt{3}$, $\sqrt[3]{5}$, or $\sqrt[4]{10}$? 22. $\sqrt{\frac{2}{3}}$, $\sqrt[3]{\frac{3}{2}}$, or $\sqrt[4]{\frac{7}{4}}$?

Multiplication of Surds.

12. Multiplication of Monomial Surds. — The product of two or more monomial surds is found by applying the principle

$$\sqrt[q]{a} \times \sqrt[q]{b} = \sqrt[q]{(ab)}.$$

Ex. **1.** $5\sqrt[3]{4} \times 2\sqrt[3]{6} = 10\sqrt[3]{24} = 20\sqrt[3]{3}$.

If the surds are of different orders, they should first be reduced to equivalent surds of the same order.

Ex. **2.** $\sqrt{a} \times \sqrt[3]{a^2} = \sqrt[6]{a^3} \times \sqrt[6]{a^4} = \sqrt[6]{a^7} = a\sqrt[6]{a}$.

Ex. **3.**

$$\begin{aligned}\sqrt[3]{(a^2b)} \times \sqrt[4]{(a^3b^2)} \times \sqrt[6]{(a^4b^5)} &= \sqrt[12]{(a^8b^4)} \times \sqrt[12]{(a^9b^6)} \times \sqrt[12]{(a^8b^{10})} \\ &= \sqrt[12]{(a^{25}b^{20})} = a^2b\sqrt[12]{(ab^8)}.\end{aligned}$$

Ex. 4.
$$\begin{aligned} 2\sqrt[3]{5} \times 3\sqrt[4]{20} \times \sqrt{10} &= 6\sqrt[3]{5} \times \sqrt[4]{(2^2 \times 5)} \times \sqrt{(2 \times 5)} \\ &= 6\sqrt[12]{5^4} \times \sqrt[12]{(2^6 \times 5^3)} \times \sqrt[12]{(2^6 \times 5^6)} \\ &= 6\sqrt[12]{(5^{13} \times 2^{12})} = 60\sqrt[12]{5}. \end{aligned}$$

When the radicands contain numerical factors it is advisable to express them as powers of the smallest possible bases, as in Ex. 4.

It is frequently desirable to introduce the coefficient of a surd under the radical sign.

Ex. 5. $3a\sqrt[3]{(2ab)} = \sqrt[3]{(27a^3)} \times \sqrt[3]{(2ab)} = \sqrt[3]{(54a^4b)}$.

13. Multiplication of Multinomial Surd Numbers. — The work may be arranged as in multiplication of rational multinomials.

Ex. Multiply $2 - 3\sqrt{2} + 5\sqrt{6}$ by $\sqrt{2} - \sqrt{3}$.

We have
$$\begin{array}{rrrr} & 2 & -\,3\sqrt{2} & +\,5\sqrt{6} \\ & \sqrt{2} & -\,\sqrt{3} & \\ \hline 2\sqrt{2} & -\,6 & +\,10\sqrt{3} & \\ -15\sqrt{2} & & -\,2\sqrt{3} & +\,3\sqrt{6} \\ \hline -13\sqrt{2} & -\,6 & +\,8\sqrt{3} & +\,3\sqrt{6} \end{array}$$

Observe that the terms of each partial product are simplified, and that similar surds are then written in the same column.

14. Conjugate Surds. — Two binomial quadratic surds which differ only in the sign of a surd term are called **Conjugate Surds**.

E.g., $\sqrt{3} + \sqrt{2}$ and $-\sqrt{3} + \sqrt{2}$; $1 - \sqrt{5}$ and $1 + \sqrt{5}$.

Either of two conjugate surds is the conjugate of the other.

The product of two conjugate surds is a rational number.

For, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$.

15. Type-Forms. — Many products are more easily obtained by using the type-forms given in Ch. VI., § 1.

Ex.
$$\begin{aligned} (\sqrt{2} + \sqrt{3})^2 &= (\sqrt{2})^2 + 2\sqrt{2} \times \sqrt{3} + (\sqrt{3})^2 \\ &= 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}. \end{aligned}$$

EXERCISES IV.

Simplify each of the following expressions :

1. $\sqrt{3} \times \sqrt{6}$.
2. $\sqrt{27} \times 3\sqrt{18}$.
3. $4\sqrt{\tfrac{3}{2}} \times 5\sqrt{\tfrac{3}{8}}$.
4. $\sqrt{\tfrac{5}{7}} \times \sqrt{\tfrac{7}{125}}$.
5. $\sqrt[3]{4} \times \sqrt[3]{2}$.
6. $2\sqrt[3]{\tfrac{25}{16}} \times \tfrac{1}{5}\sqrt[3]{\tfrac{2}{5}}$.

7. $\sqrt[3]{(a^2b)} \times \sqrt[3]{(ab^2)}$. 8. $\sqrt[4]{8} \times \sqrt[4]{10}$. 9. $9\sqrt[4]{54} \times 3\sqrt[4]{24}$.

10. $\sqrt{2} \times 2\sqrt[4]{4}$. 11. $\sqrt{\frac{1}{2}} \times \sqrt[6]{12}$. 12. $\sqrt[4]{54} \times \sqrt[6]{486}$.

13. $\sqrt{\frac{10}{11}} \times \sqrt[3]{\frac{11}{10}}$. 14. $\sqrt[3]{12} \times \sqrt[4]{6}$. 15. $\sqrt[3]{\frac{5}{6}} \times \sqrt[4]{\frac{18}{25}}$.

16. $\sqrt[4]{\frac{27}{64}} \times \sqrt[3]{\frac{81}{16}}$. 17. $\sqrt[5]{\frac{7}{15}} \times \sqrt[6]{\frac{15}{14}}$. 18. $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{3}} \times \sqrt[9]{3}$.

19. $\sqrt[5]{54} \times 3\sqrt{6} \times 5\sqrt[3]{2}$. 20. $\sqrt{10} \times \sqrt[3]{100} \times \sqrt[4]{500}$.

21. $\sqrt[3]{12} \times \sqrt[4]{108} \times \sqrt[6]{486}$. 22. $12\sqrt[4]{14} \times \sqrt{2\frac{1}{7}} \times \sqrt[8]{\frac{49}{300}}$.

23. $\sqrt{(x^3 + x)} \times \sqrt{(ax + a)}$. 24. $\sqrt{(12x^3 - 12x)} \times \sqrt{(3x^2 - 3)}$.

25. $\sqrt{(x^2 - x)} \times \sqrt{(x^4 + x^3)}$. 26. $\sqrt{(ax + a)} \times \sqrt{(bx + b)}$.

27. $\sqrt{(a^2 - b^2)} \times \sqrt{\frac{a + b}{a - b}}$. 28. $\sqrt{(6x^2 - 6)} \times \sqrt{\frac{3x - 3}{2x + 2}}$.

29. $\dfrac{x^3 - 8z^3}{\sqrt{(x^3 + 2x^2z + 4xz^2)}} \times \dfrac{x^2}{x - 2z}\sqrt{\dfrac{xz}{x^2 + 2xz + 4z^2}}$.

30. $\dfrac{a^2 - 25m^2}{4a^2 - 9m^2}\sqrt[3]{\dfrac{4a^2 - 12am + 9m^2}{a^2 + 10am + 25m^2}} \times \sqrt[3]{\dfrac{4a^2 - 12am + 9m^2}{a^2 + 5am}}$.

31. $(\sqrt{2} - \sqrt{3} + \sqrt{18})\sqrt{2}$. 32. $(4\sqrt[3]{9} - 2\frac{3}{4}\sqrt[3]{36}) \times 2\sqrt[3]{30}$.

33. $(\sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2})\sqrt[6]{\frac{1}{2}}$. 34. $(3 + \sqrt{5})(2 - \sqrt{5})$.

35. $(9 - 7\sqrt{13})(5 - 6\sqrt{13})$. 36. $(13 - \sqrt{5})(7 + 3\sqrt{5})$.

37. $(5 + \sqrt[3]{4} - 2\sqrt[4]{5})(\sqrt{6} + \sqrt{5})$. 38. $(2\sqrt{3} + \sqrt[3]{2})(2\sqrt{3} - \sqrt[3]{4})$.

Find the value of each of the following powers:

39. $(\sqrt{7})^3$. 40. $(2\sqrt{3})^4$. 41. $(\sqrt{x})^3$. 42. $(\sqrt[3]{ab})^2$.

43. $(5\sqrt[3]{4})^5$. 44. $(\sqrt[4]{xy})^2$. 45. $(\frac{1}{2}\sqrt{6ab})^3$. 46. $(2a\sqrt[3]{3b})^6$.

47. $(\sqrt[6]{a^2b^2})^3$. 48. $(\sqrt[6]{8ab})^3$. 49. $\left[\dfrac{\sqrt[5]{(a^4b)}}{\sqrt[4]{(a^3b)}}\right]^{10}$. 50. $\left[\dfrac{\sqrt[4]{(a^3x^2)}}{\sqrt[3]{(ax^2)}}\right]^2$.

Find the value of each of the following expressions, without performing the actual multiplication:

51. $(\sqrt{5} - \sqrt{10})^2$. 52. $(\frac{1}{2} + 2\sqrt{2})^2$. 53. $(\sqrt[4]{8} - \sqrt[4]{2})^2$.

54. $(\sqrt{6} - \sqrt[4]{40})^2$. 55. $(\sqrt{3} - \sqrt{6})^2$. 56. $(\sqrt{6} - 2\sqrt[3]{2})^3$.

57. $(1 + \sqrt{2} - \sqrt{3})^2$. 58. $(\sqrt{2} + \sqrt{3} + 1)^2$.

59. $\sqrt[4]{(5 + 2\sqrt{6})} \times \sqrt{(3 - \sqrt{6})}$. 60. $(8 - 3\sqrt{7})(8 + 3\sqrt{7})$.

61. $\sqrt[3]{(2 + \sqrt{12})}\sqrt[3]{(2 - \sqrt{12})}$. 62. $(\sqrt{3 - \sqrt{5}} + \sqrt{3 + \sqrt{5}})^2$.

63. $(\sqrt{2ab} + \sqrt{3ab})^2$. 64. $(n - \sqrt{1 - n^2})^2$.

65. $(\sqrt{a + x} + \sqrt{a - x})^2$. 66. $(\sqrt[3]{2a^2} + \sqrt[3]{2a})^3$. 67. $(x + 2\sqrt{x^2 - 1})^3$.

68. $(\sqrt{a + b} + \sqrt{a} - \sqrt{b})(\sqrt{a + b} - \sqrt{a} + \sqrt{b})$.

69. $[\sqrt{\sqrt{(a + b)} + \sqrt{(a - b)}} - \sqrt{\sqrt{(a + b)} - \sqrt{(a - b)}}]^2$.

70. $\left(x + \dfrac{p}{2} + \sqrt{\dfrac{p^2}{4} - q}\right)\left(x + \dfrac{p}{2} - \sqrt{\dfrac{p^2}{4} - q}\right)$.

In each of the following expressions introduce the coefficient under the radical sign:

71. $7\sqrt{3}$. **72.** $\frac{1}{2}\sqrt{2}$. **73.** $\frac{1}{2}\sqrt[3]{4}$. **74.** $\frac{3}{4}\sqrt[3]{\frac{16}{9}}$.

75. $2a\sqrt{a}$. **76.** $5x^2\sqrt{(3xy)}$. **77.** $ab\sqrt{\frac{1}{ab}}$. **78.** $4a^2b\sqrt[3]{(2a)}$.

79. $a\sqrt[n]{a}$. **80.** $a^2b\sqrt[n-1]{(ab)}$. **81.** $a^{n+1}\sqrt{a^{n-2}}$. **82.** $x^ny^m\sqrt[3]{(x^my^n)}$.

83. $(a+b)\sqrt{\frac{ab}{a^2+2ab+b^2}}$. **84.** $(m-n)\sqrt{\frac{m+n}{m-n}}$.

Division of Surds.

16. Division of Monomial Surds. — The quotient of one monomial surd divided by another is obtained by applying the principle

$$\frac{\sqrt[q]{a}}{\sqrt[q]{b}}=\sqrt[q]{\frac{a}{b}}.$$

Ex. 1. $\frac{\sqrt{8}}{\sqrt{2}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2.$

If the surds are of different orders, they should first be reduced to equivalent surds of the same order.

Ex. 2. $\frac{\sqrt[3]{(4a^2)}}{\sqrt[2]{(3a)}}=\frac{\sqrt[6]{(16a^4)}}{\sqrt[6]{(27a^3)}}=\sqrt[6]{\frac{16a}{27}}=\frac{1}{3}\sqrt[6]{(432a)}.$

17. Division of Multinomial Surd Numbers. — It is better to write the quotient of one multinomial surd number by another as a fraction, and then to simplify this fraction by the method to be given in Art. 26. But if the divisor is a monomial, the work proceeds as follows:

$$(\sqrt{72}+\sqrt{32}-4)\div 2\sqrt{2}=\frac{\sqrt{36}}{2}+\frac{\sqrt{16}}{2}-\frac{2}{\sqrt{2}}$$
$$=3+2-\sqrt{2}=5-\sqrt{2}.$$

18. Type-Forms. — Many quotients are more easily obtained by using the type-forms given in Ch. VI., § 2.

Ex. $(\sqrt[3]{a^2}-\sqrt[3]{b^2})\div(\sqrt[3]{a}-\sqrt[3]{b})=[(\sqrt[3]{a})^2-(\sqrt[3]{b})^2]\div(\sqrt[3]{a}-\sqrt[3]{b})$
$=\sqrt[3]{a}+\sqrt[3]{b}.$

EXERCISES V.

Simplify each of the following expressions:

1. $3\sqrt{2} \div 2\sqrt{3}$.
2. $\sqrt{60} \div \sqrt{5}$.
3. $\sqrt{15} \div \sqrt{\frac{3}{5}}$.
4. $\sqrt{21} \div \sqrt{\frac{7}{3}}$.
5. $6 \div \sqrt{3}$.
6. $20 \div 3\sqrt{10}$.
7. $10 \div \sqrt[3]{5}$.
8. $15 \div \sqrt[4]{3}$.
9. $9\sqrt[3]{7} \div 2\sqrt[3]{21}$.
10. $2\sqrt[3]{6} \div \sqrt[6]{2}$.
11. $6\sqrt{2} \div \sqrt[3]{9}$.
12. $\sqrt[3]{20} \div 3\sqrt[9]{16}$.
13. $(4 + \sqrt{6} - 5\sqrt{14}) \div 2\sqrt{2}$.
14. $(3\sqrt{10} - 4\sqrt{15} + 5) \div \sqrt{5}$.
15. $(\sqrt{2} - 3\sqrt[4]{4}) \div \sqrt[4]{2}$.
16. $(\sqrt[3]{3} - 3\sqrt[6]{6}) \div \sqrt[6]{3}$.
17. $\sqrt{x} \div \sqrt[3]{x}$.
18. $\sqrt{x^3} \div \sqrt[3]{x^2}$.
19. $\sqrt{x} \div \sqrt[4]{x}$.
20. $\sqrt[3]{x^2} \div \sqrt[4]{x^3}$.
21. $\sqrt{(a^2x)} \div \sqrt{x}$.
22. $\sqrt[3]{x^2} \div n\sqrt{x}$.
23. $\sqrt{(14\,ab)} \div \sqrt[3]{(28\,a^2b^2)}$.
24. $\sqrt[3]{(15\,x^2y)} \div \sqrt[4]{(25\,xy^2)}$.
25. $2\,a^2\sqrt{n} \div 5\sqrt[3]{(4\,n)}$.
26. $x\sqrt[2n]{x^{n-3}} \div n\sqrt[n]{x^{n-4}}$.

Simplify each of the following expressions, without performing the actual division:

27. $(1 - x) \div (1 - \sqrt{x})$.
28. $(ax - bx) \div (\sqrt{a} - \sqrt{b})$.
29. $\left(1 - \frac{1}{x}\right) \div (1 + \sqrt{x})$.
30. $\left(\frac{a}{b} - \frac{x}{y}\right) \div \left(\sqrt{\frac{a}{x}} + \sqrt{\frac{b}{y}}\right)$.
31. $(a\sqrt{a} + b\sqrt{b}) \div (\sqrt{a} + \sqrt{b})$.
32. $(x\sqrt[3]{x} - y\sqrt[3]{y}) \div (\sqrt[3]{x} - \sqrt[3]{y})$.

Surd Factors.

19. From the identity

$$(mx + n)^2 = m^2x^2 + 2\,mnx + n^2$$

we infer:

If a trinomial, arranged to descending powers of a letter, say x, be the square of a binomial, the third term is equal to the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2; that is,

$$n^2 = \left(\frac{2\,mn}{2\,m}\right)^2.$$

Consequently, if to any binomial of the form $m^2x^2 + 2\,mnx$ the term $\left(\frac{2\,mn}{2\,m}\right)^2$, $= n^2$, be added, the resulting trinomial will be the square of a binomial.

This step is called *completing the square.*

E.g., if to $9\,x^2 + 5\,x$ we add $\left(\frac{5}{2 \times 3}\right)^2$, $= \frac{25}{36}$,

we have $\qquad 9\,x^2 + 5\,x + \frac{25}{36}$, $= (3\,x + \frac{5}{6})^2$.

20. An expression of the second degree in a letter of arrangement, say x, can be transformed into the difference of two squares, and hence be factored.

Ex. Factor $25x^2 + 13x + 1$.

To transform $25x^2 + 13x + 1$ into the difference of two squares, we first complete $25x^2 + 13x$ to the square of a binomial by adding $(\frac{13}{2 \times 5})^2, = \frac{169}{100}$; and, in order that the value of the given expression may remain unchanged, we also subtract $\frac{169}{100}$ from it. We then have

$$
\begin{aligned}
25x^2 + 13x + 1 &= 25x^2 + 13x + \tfrac{169}{100} - \tfrac{169}{100} + 1 \\
&= (5x + \tfrac{13}{10})^2 - (\tfrac{\sqrt{69}}{10})^2 \\
&= (5x + \tfrac{13}{10} + \tfrac{1}{10}\sqrt{69})(5x + \tfrac{13}{10} - \tfrac{1}{10}\sqrt{69}).
\end{aligned}
$$

21. *If the coefficient of x^2 in the expression to be factored be 1, the term to be added to complete the square is evidently the square of half the coefficient of x.*

Ex. 1. Factor $x^2 - 5x - 1$.

We have
$$
\begin{aligned}
x^2 - 5x - 1 &= x^2 - 5x + (\tfrac{5}{2})^2 - (\tfrac{5}{2})^2 - 1 \\
&= (x - \tfrac{5}{2})^2 - (\tfrac{\sqrt{29}}{2})^2 \\
&= (x - \tfrac{5}{2} + \tfrac{1}{2}\sqrt{29})(x - \tfrac{5}{2} - \tfrac{1}{2}\sqrt{29}).
\end{aligned}
$$

Ex. 2. Factor $-3x^2 + 4xy + 2y^2$.

Since the coefficient of x^2 is not the square of a rational number, the work is simplified by first taking out the factor -3. We then have

$$-3x^2 + 4xy + 2y^2 = -3(x^2 - \tfrac{4}{3}xy - \tfrac{2}{3}y^2).$$

Completing $x^2 - \frac{4}{3}xy$ to the square of a binomial by adding $(\frac{2}{3}y)^2, = \frac{4}{9}y^2$, to the expression *within the parentheses*, and also subtracting $\frac{4}{9}y^2$ from it, we obtain

$$
\begin{aligned}
-3x^2 + 4xy + 2y^2 &= -3(x^2 - \tfrac{4}{3}xy + \tfrac{4}{9}y^2 - \tfrac{10}{9}y^2) \\
&= -3[(x - \tfrac{2}{3}y)^2 - (\tfrac{\sqrt{10}}{3}y)^2] \\
&= -3(x - \tfrac{2}{3}y + \tfrac{\sqrt{10}}{3}y)(x - \tfrac{2}{3}y - \tfrac{\sqrt{10}}{3}y).
\end{aligned}
$$

This method can of course be applied when the factors are rational, but the methods given in Ch. VIII., § 1, Arts. 9–13, are, as a rule, to be preferred.

EXERCISES VI.

Resolve $x - 1$ into two factors, one of which is

1. $\sqrt{x} + 1$. 2. $\sqrt[3]{x} - 1$. 3. $\sqrt[4]{x} + 1$. 4. $\sqrt[5]{x} - 1$.

Factor each of the following expressions:

5. $x^2 - 2x - 11$. 6. $166 + 6x - x^2$. 7. $4x^2 - 4xy - 17y^2$.
8. $3 + 2x - 11x^2$. 9. $x^2 - 2mx - 1$. 10. $x^2 - 2ax + a^2 - b^2$.
11. $ax^2 + bxy + cy^2$. 12. $m^2x^2 - 4mx + 4 - nm^2$.

Rationalization.

22. To *rationalize* a surd expression is to free it from irrational numbers.

Thus, $\sqrt[3]{4}$ is rationalized by multiplying it by $\sqrt[3]{2}$, since $\sqrt[3]{4} \times \sqrt[3]{2} = \sqrt[3]{8} = 2$.

A **Rationalizing Factor** for an irrational expression is an expression which, multiplying the irrational expression, gives a rational product.

E.g., $\sqrt[3]{2}$ is a rationalizing factor for $\sqrt[3]{4}$, and *vice versa.*

23. A rationalizing factor for a monomial surd number is easily determined by inspection.

Ex. 1. A rationalizing factor for $\sqrt[q]{a^p}$ is $\sqrt[q]{a^{q-p}}$, and *vice versa.*

Ex. 2 A rationalizing factor for $\sqrt[5]{(a^3b)}$ is $\sqrt[5]{(a^2b^4)}$, and *vice versa.*

24. A rationalizing factor for a binomial quadratic surd is its conjugate (Art. 14).

Ex. 1. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3}) = 2 - 3 = -1$.

Either of the given binomial surds is a rationalizing factor for the other.

25. Rationalizing factors for a trinomial quadratic surd can be found by reference to the following identity:

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} - \sqrt{c}) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc.$$

The rationalizing factors for any one of the four factors on the left are the other three.

E.g., $\sqrt{3}-\sqrt{5}+\sqrt{7}$ has the rationalizing factors

$$(\sqrt{3}+\sqrt{5}+\sqrt{7})(\sqrt{3}+\sqrt{5}-\sqrt{7})(\sqrt{3}-\sqrt{5}-\sqrt{7}).$$

In rationalizing numerical examples, the work is simplified if, after multiplying by one of the three rationalizing factors, the product, which is a binomial, be multiplied by its conjugate. Thus,

$$\begin{aligned}(\sqrt{3}-\sqrt{5}+\sqrt{7})(\sqrt{3}-\sqrt{5}-\sqrt{7}) &= 3+5-2\sqrt{15}-7\\ &= 1-2\sqrt{15}.\end{aligned}$$

Therefore the second rationalizing factor is $1+2\sqrt{15}$.

EXERCISES VII.

Find the expressions which will rationalize the following:

1. $\sqrt{3}$. 2. $\sqrt[3]{4}$. 3. $\sqrt[3]{7}$. 4. $\sqrt[4]{12}$. 5. $\sqrt[5]{16}$.
6. $\sqrt{a}$. 7. $\sqrt[3]{(ab^2)}$. 8. $\sqrt[4]{(a^3x)}$. 9. $\sqrt[11]{(a^9b^2)}$. 10. $\sqrt[n]{(a^2b^{n-2})}$.
11. $3+\sqrt{2}$. 12. $2\sqrt{14}-\sqrt{3}$. 13. $\sqrt{5}+\sqrt{3}-\sqrt{2}$.
14. $\sqrt{(a^2-1)}-\sqrt{(a^2+1)}$. 15. $\sqrt{(x^2+y^2)}-\sqrt{(2\,xy)}$.

Reduction of a Fraction with an Irrational Denominator to an Equivalent Fraction with a Rational Denominator.

26. Multiply both numerator and denominator of the given fraction by the rationalizing factor for the denominator.

Ex. 1. $$\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{3}.$$

Ex. 2. $$\frac{1}{1+\sqrt{2}}=\frac{1}{1+\sqrt{2}}\times\frac{1-\sqrt{2}}{1-\sqrt{2}}=\frac{1-\sqrt{2}}{1-2}=-1+\sqrt{2}.$$

Ex. 3.

$$\begin{aligned}\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}} &= \frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}\times\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}\\ &= \frac{1+x+2\sqrt{(1-x^2)}+1-x}{(1+x)-(1-x)}=\frac{1+\sqrt{(1-x^2)}}{x}.\end{aligned}$$

Ex. 4.

$$\begin{aligned}\frac{3\sqrt{2}-1}{2\sqrt{2}-\sqrt{3}+\sqrt{6}} &= \frac{3\sqrt{2}-1}{2\sqrt{2}-\sqrt{3}+\sqrt{6}}\times\frac{2\sqrt{2}-\sqrt{3}-\sqrt{6}}{2\sqrt{2}-\sqrt{3}-\sqrt{6}}\\ &= \frac{12-2\sqrt{6}-5\sqrt{3}-2\sqrt{2}}{5-4\sqrt{6}}\times\frac{5+4\sqrt{6}}{5+4\sqrt{6}}\\ &= \frac{12+38\sqrt{6}-41\sqrt{3}-70\sqrt{2}}{-71}.\end{aligned}$$

EXERCISES VIII.

Change each of the following fractions into an equivalent fraction with a rational denominator:

1. $\dfrac{1}{\sqrt{2}}$. 2. $\dfrac{12}{5\sqrt{3}}$. 3. $\dfrac{8}{3\sqrt[3]{4}}$. 4. $\dfrac{10}{3\sqrt[4]{25}}$. 5. $\dfrac{14}{2\sqrt[5]{16}}$.

6. $\dfrac{x}{\sqrt{x}}$. 7. $\dfrac{x}{\sqrt[3]{x^2}}$. 8. $\dfrac{a}{\sqrt[n]{a^4}}$. 9. $\dfrac{1}{2-\sqrt{3}}$. 10. $\dfrac{12}{5-\sqrt{21}}$.

11. $\dfrac{\sqrt{(2m)}+3n}{\sqrt{(2m)}-3n}$. 12. $\dfrac{1+\sqrt{2}}{2-\sqrt{2}}$. 13. $\dfrac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-\sqrt{18}}$.

14. $\dfrac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$. 15. $\dfrac{3a^2-2a\sqrt{(ab)}}{3a+2\sqrt{(ab)}}$. 16. $\dfrac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$.

17. $\dfrac{12}{\sqrt{6}-\sqrt{10}+2}$. 18. $\dfrac{2-\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$. 19. $\dfrac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$.

20. $\dfrac{(a^2+b)+a\sqrt{(a^2+b)}}{a+\sqrt{(a^2+b)}}$. 21. $\dfrac{(n+1)+\sqrt{(n^2-1)}}{(n+1)-\sqrt{(n^2-1)}}$.

22. $\dfrac{a+\sqrt{(a^2-4a)}}{a-2+\sqrt{(a^2-4a)}}$. 23. $\dfrac{\sqrt{a}+\sqrt{(a+x)}}{\sqrt{a}+\sqrt{x}+\sqrt{(a+x)}}$.

Properties of Quadratic Surds.

27. *The product and the quotient of two like quadratic surds are rational.*

For $m\sqrt{x} \times n\sqrt{x} = mn\sqrt{x^2} = mnx$, and $\dfrac{m\sqrt{x}}{n\sqrt{x}} = \dfrac{m}{n}$.

E.g., $3\sqrt{2} \times 5\sqrt{2} = 15\sqrt{4} = 30$; $\dfrac{4\sqrt{3}}{2\sqrt{3}} = 2$.

28. *If the product, or the quotient, of two quadratic surds be rational, they must be like surds.*

Let $\sqrt{x} \times \sqrt{y} = R$, a rational number.

Then $$\sqrt{x} = \frac{R}{\sqrt{y}} = \frac{R}{y}\sqrt{y}.$$

Since $\sqrt{x}$ is a rational multiple of $\sqrt{y}$, therefore $\sqrt{x}$ and $\sqrt{y}$ must be like surds.

In like manner the principle can be proved for the quotient.

29. *The product and the quotient of two unlike quadratic surds are irrational.*

For, by Art. 28, whenever the product of two quadratic surds is rational, they must be like surds.

E.g., $\sqrt{2} \times \sqrt{3} = \sqrt{6}$; $\frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$.

30. *A quadratic surd cannot be equal to the sum of a rational number and another quadratic surd;* or

$$\sqrt{a} \neq b + \sqrt{c},$$

wherein $\sqrt{a}$ and $\sqrt{c}$ are surds, and b is rational.

For if $\sqrt{a} = b + \sqrt{c}$, then $a = b^2 + c + 2b\sqrt{c}$.

Solving the last equation for $\sqrt{c}$, we obtain

$$\sqrt{c} = \frac{a - b^2 - c}{2b}.$$

This equation asserts that $\sqrt{c}$, an irrational number, is equal to $\frac{a - b^2 - c}{2b}$, a rational number. This is a contradiction of terms, and therefore the hypothesis $\sqrt{a} = b + \sqrt{c}$ is untenable.

31. *If* $$a + \sqrt{b} = x + \sqrt{y}, \qquad (1)$$

wherein $\sqrt{b}$ and $\sqrt{y}$ are surds, and a and x are rational, then $a = x$ and $b = y$.

For if $a \neq x$, let $a = x + m$, wherein $m \neq 0$.

Then (1) becomes $x + m + \sqrt{b} = x + \sqrt{y}$, or $m + \sqrt{b} = \sqrt{y}$. (2)

But by the preceding article (2) is untenable, unless $m = 0$.

Therefore $a = x$, and hence $\sqrt{b} = \sqrt{y}$, or $b = y$.

32. *If* $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$, then $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}$.

From $$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y},$$

we obtain $$a + \sqrt{b} = x + y + 2\sqrt{(xy)}.$$

Whence, by Art. 31, $$a = x + y, \qquad (1)$$

and $$\sqrt{b} = 2\sqrt{(xy)}. \qquad (2)$$

Subtracting (2) from (1),

$$a - \sqrt{b} = x + y - 2\sqrt{(xy)}. \qquad (3)$$

Therefore $$\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

Evolution of Surd Expressions.

33. A root of a monomial surd number is found by applying the principle

$$\sqrt[q]{\sqrt[r]{a}}=\sqrt[qr]{a}. \quad \text{[Ch. XVI., § 1, Art. 13 (v.}$$

Ex. 1. $$\sqrt[4]{\sqrt[3]{5}}=\sqrt[12]{5}.$$

It is important to notice that $\sqrt[q]{\sqrt[r]{a}}=\sqrt[r]{\sqrt[q]{a}}$.

Ex. 2. $$\sqrt[3]{\sqrt[5]{(8x^3)}}=\sqrt[5]{\sqrt[3]{(8x^3)}}=\sqrt[5]{(2x)}.$$

EXERCISES IX.

Simplify each of the following expressions:

1. $\sqrt[4]{\sqrt[3]{a^8}}$. 2. $\sqrt[6]{\sqrt[3]{a^2}}$. 3. $\sqrt[3]{\sqrt[5]{(-x^3)}}$.

4. $\sqrt[3]{\sqrt[4]{(a^9x^{12})}}$. 5. $\sqrt[4]{(2a\sqrt[3]{a^2})}$. 6. $\sqrt[3]{(a\sqrt{a})}$.

7. $\sqrt[m]{\sqrt[n]{a^m}}$. 8. $\sqrt{\sqrt[3]{(\frac{25}{49}a^2b^6c^8)}}$. 9. $\sqrt[5]{(a^2\sqrt{a})}$.

10. $\sqrt{\frac{2}{\sqrt[3]{2}}}$. 11. $\sqrt[x-1]{\frac{a}{\sqrt[x]{a}}}$. 12. $\sqrt[3]{\frac{a^2}{\sqrt{a}}}$.

13. $2\sqrt{\{2\sqrt{[2\sqrt{(2\sqrt{2})}]}\}}$. 14. $a\sqrt{a\sqrt{\{a\sqrt{[a\sqrt{(a\sqrt{a})}]}\}}}$.

15. $2\sqrt{\sqrt[6]{7}}+3\sqrt{\sqrt[3]{\sqrt{7}}}-3\sqrt[4]{\sqrt[3]{7}}-\sqrt[3]{\sqrt{\sqrt{7}}}$.

16. $\sqrt[2m]{\sqrt[3n]{a^5}}\times\sqrt[6m]{\sqrt[n]{a^8}}\times\sqrt[m]{\sqrt[6n]{a^9}}\times\sqrt[6m]{\sqrt[n]{a}}$.

Square Roots of Simple Binomial Surds.

34. Ex. 1. Find a square root of $3+2\sqrt{2}$.

Let $$\sqrt{(3+2\sqrt{2})}=\sqrt{x}+\sqrt{y}. \qquad (1)$$

Then, by Art. 32, $$\sqrt{(3-2\sqrt{2})}=\sqrt{x}-\sqrt{y}. \qquad (2)$$

Multiplying (1) by (2), $\sqrt{(9-8)}=x-y$,

or $$x-y=1. \qquad (3)$$

Squaring (1), $3+2\sqrt{2}=x+y+2\sqrt{(xy)}$;

whence, by Art. 31, $$x+y=3. \qquad (4)$$

Solving (3) and (4), we have $x=2$, $y=1$.

Therefore $\sqrt{(3+2\sqrt{2})}=\sqrt{2}+\sqrt{1}=\sqrt{2}+1$.

This example could have been solved by inspection. We change $3+2\sqrt{2}$ into the form

$$m+2\sqrt{(mn)}+n=(\sqrt{m}+\sqrt{n})^2.$$

We then have

$$\sqrt{(3+2\sqrt{2})}=\sqrt{(2+2\sqrt{2}+1)}=\sqrt{(\sqrt{2}+1)^2}=\sqrt{2}+1.$$

Ex. 2. Solve, by inspection, $\sqrt{(21-3\sqrt{24})}$.

We have
$$\begin{aligned}\sqrt{(21-3\sqrt{24})}&=\sqrt{(21-2\sqrt{54})}\\&=\sqrt{(18-2\sqrt{54}+3)}\\&=\sqrt{(\sqrt{18}-\sqrt{3})^2}\\&=\sqrt{18}-\sqrt{3}=3\sqrt{2}-\sqrt{3}.\end{aligned}$$

In solving by inspection, first write the surd term of the given binomial surd in the form $2\sqrt{(mn)}$, *as* $3\sqrt{24}=2\sqrt{54}$.

Then find by inspection two numbers whose sum is equal to the rational term of the given binomial surd, and whose product is equal to mn.

EXERCISES X.

Find a square root of each of the following expressions:

1. $7+\sqrt{48}$. **2.** $5-\sqrt{24}$. **3.** $2+\sqrt{3}$.
4. $1\frac{1}{2}+\sqrt{2}$. **5.** $3-\sqrt{5}$. **6.** $6+\sqrt{11}$.
7. $8-\sqrt{28}$. **8.** $6+4\sqrt{2}$. **9.** $7+2\sqrt{10}$.
10. $11-6\sqrt{2}$. **11.** $11+4\sqrt{7}$. **12.** $30-10\sqrt{5}$.
13. $\frac{5}{7}+\frac{1}{7}\sqrt{21}$. **14.** $\frac{9}{11}-\frac{4}{11}\sqrt{2}$. **15.** $\frac{21}{22}-\frac{2}{11}\sqrt{5}$.
16. $4a+2\sqrt{(4a^2-b^2)}$. **17.** $n-2\sqrt{(n-1)}$.
18. $10n^2+1-6n\sqrt{(n^2+1)}$. **19.** $a-x-2\sqrt{(a-x-1)}$.

Approximate Values of Surd Numbers.

35. An approximate value of a surd number can be found to any degree of accuracy by the methods given in Ch. XVI.

Ex. 1. Find an approximate value of $\sqrt{2}$ correct to three decimal places. The work proceeds as follows:

$$\begin{array}{r|l}
2.00'00'00'00 & \underline{1.4142} \\
\underline{1} & 2 \\
1\,00 & \\
\underline{96} & \underline{24} \\
4\,00 & \\
\underline{2\,81} & \underline{281} \\
1\,19\,00 & \\
\underline{1\,12\,96} & \underline{2824} \\
6\,04\,00 & 2828
\end{array}$$

The work is simplified by neglecting the decimal point, writing it only in the result. It is necessary to find the root to four decimal places in order to determine whether to take the figure found in the third place or the next greater figure, according to the well-known principle of Arithmetic.

Ex. 2. Find the value of $\sqrt[3]{(1-x)}$ to three terms.

The work proceeds as follows:

$$\begin{array}{l|l}
1-x & \underline{1-\frac{1}{3}x-\frac{1}{9}x^2} \\
\underline{1} & 3\times 1^2=3 \\
\quad -x & 3\times 1^2+3\times 1\times(-\frac{1}{3}x)+(-\frac{1}{3}x)^2=3-x+\frac{1}{9}x^2 \\
\quad -x+\frac{1}{3}x^2-\frac{1}{27}x^3 & \\
\hline
\qquad -\frac{1}{3}x^2+\frac{1}{27}x^3 &
\end{array}$$

An approximate value of a fractional surd is obtained most simply by rationalizing its denominator, then finding the required root of the numerator of the resulting fraction, and dividing this value by the denominator.

Ex. 3. Find an approximate value of $\sqrt{\frac{1}{2}}$ correct to three decimal places.

We have $\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{2}$, and $\sqrt{2}=1.4142+\cdots$.

Therefore $\sqrt{\frac{1}{2}}=.707$, correct to three places of decimals.

EXERCISES XI.

Find by inspection the square root of each of the following expressions:

1. $a^2+2a\sqrt{b}+b$.
2. $4a+9x-12\sqrt{(ax)}$.
3. $9+6\sqrt[3]{3}+\sqrt[3]{9}$.
4. $\sqrt[3]{5}+2\sqrt[3]{2}+2\sqrt[6]{80}$.

Find by inspection the cube root of each of the following expressions:

5. $x\sqrt{x}+3\sqrt{x}-3x-1$.
6. $4n+12n\sqrt[3]{n^2}+12n\sqrt[3]{n}+4n^2$.
7. $8x^3+66x^2+33x-36x^2\sqrt{x}-63x\sqrt{x}-9\sqrt{x}+1$.

Find an approximate value of each of the following expressions, correct to four figures:

8. $\sqrt{8}$.
9. $\frac{1}{2}\sqrt{2.5}$.
10. $\sqrt{2}$.
11. $\frac{2}{3}\sqrt{1.25}$.
12. $\sqrt{345.06}$.
13. $\sqrt{10862.321}$.
14. $\sqrt{54.0001}$.
15. $\dfrac{2}{\sqrt{5}}$.
16. $\dfrac{3}{\sqrt{8}}$.
17. $\dfrac{1}{2\sqrt[3]{4}}$.
18. $\dfrac{5}{\sqrt{75}}$.
19. $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$.
20. $\dfrac{3+2\sqrt{7}}{5-4\sqrt{11}}$.
21. $\dfrac{\sqrt{17}}{\sqrt{2.5}+\sqrt{6}}$.

Find an approximate value of each of the following expressions, to nclude four terms:

22. $\sqrt{(1-x)}$. 23. $\sqrt{(a^2+b^2)}$. 24. $\sqrt{(x^2-xy+y^2)}$.
25. $\sqrt[3]{(1+x^3)}$. 26. $\sqrt[3]{(a^3-b^3)}$. 27. $\sqrt[3]{(x^3+x^2y+xy^2+y^3)}$.

EXERCISES XII.

MISCELLANEOUS EXAMPLES.

1. Simplify $2\sqrt{3+\sqrt{5-\sqrt{13+4\sqrt{3}}}}$.

In each of the following expressions make the indicated substitution and simplify the result:

2. In $\dfrac{1+a}{1+\sqrt{(1+a)}}+\dfrac{1-a}{1-\sqrt{(1-a)}}$, let $a=\frac{1}{2}\sqrt{3}$.

3. In $\dfrac{x}{\sqrt{(1-x^2)}}+\dfrac{\sqrt{(1-x^2)}}{x}$, let $x=\sqrt{\dfrac{m-\sqrt{(m^2-4)}}{2m}}$.

4. In $2[ab-\sqrt{(a^2-1)}\sqrt{(b^2-1)}]$, let $2a=x+\dfrac{1}{x}$ and $2b=y+\dfrac{1}{y}$.

5. In $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$, let $x=\dfrac{2ab}{b^2+1}$.

6. In x^2+y^2+xy, let $x=\frac{1}{2}[\sqrt{(a+b)}+\sqrt{(a-3b)}]$
and $y=\frac{1}{2}[\sqrt{(a+b)}-\sqrt{(a-3b)}]$.

7. In $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$, let $x=\sqrt{\dfrac{a-1}{a+1}}$.

8. In $\dfrac{2a\sqrt{(1+x^2)}}{x+\sqrt{(1+x^2)}}$, let $x=\frac{1}{2}\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)$.

9. In $x\sqrt{(1-x^2)}$, let $x=\sqrt{\dfrac{1+\sqrt{(1-4a^2)}}{2}}$.

10. In $x^3+3ax+2b$, let $x=\sqrt[3]{[-b+\sqrt{(a^3+b^2)}]}+\sqrt[3]{[-b-\sqrt{(a^3+b^2)}]}$.

11. Prove the following identity:

$$\sqrt{[2a^2-b^2+2a\sqrt{(a^2-b^2)}]}-\sqrt{[a^2-2b\sqrt{(a^2-b^2)}]}=a+b.$$

12. Simplify $\dfrac{a+\sqrt{b}}{\left[\sqrt{\dfrac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\dfrac{a-\sqrt{(a^2-b)}}{2}}\right]^2}$.

13. Simplify $\dfrac{a^3-3a+(a^2-1)\sqrt{(a^2-4)}-2}{a^3-3a+(a^2-1)\sqrt{(a^2-4)}+2}$.

CHAPTER XX.

IMAGINARY AND COMPLEX NUMBERS.

1. Since even powers of both positive and negative numbers are *positive*, even roots of negative numbers cannot be expressed in terms of numbers as yet comprised in the number system.

E.g, since $(\pm 4)^2 = 16$, the $\sqrt{-16}$ cannot be expressed as a positive or as a negative number.

It is therefore necessary either to exclude such roots from our consideration or again to enlarge our ideas of number. The latter alternative is in accordance with the generalizing spirit of Algebra.

We therefore assume that $\sqrt{-1}$, *and in general* $\sqrt[2n]{-a}$, *are numbers, and include them in the number system.*

2. These new numbers are defined by the relations

$$(\sqrt{-1})^2 = -1, \text{ and in general } (\sqrt[2n]{-a})^{2n} = -a.$$

Imaginary Numbers.

3. The square root of a negative number is called an **Imaginary Number**; as $\sqrt{-3}$, $\sqrt{-8}$.

The study of these numbers is simplified by first considering the properties of $\sqrt{-1}$, which is taken as the **Imaginary Unit.**[*] This new unit is commonly designated by the letter i, and its opposite by $-i$.

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[*] The designation, *imaginary*, is unfortunate, since, as will be shown in **Part II., Text-Book of Algebra**, such numbers are no more imaginary (in the ordinary meaning of the word) than common fractions or negative numbers. Dr. George Bruce Halsted, Professor of Mathematics in the University of Texas, has suggested **Neomon** for the *imaginary unit*, and **Neomonic** for *imaginary*.

We then have by definition

$$(\pm i)^2 = -1.$$

For the sake of distinction all numbers, rational and irrational, which have been used hitherto in this book are called **Real Numbers.**

4. Multiples and Fractional Parts of the Imaginary Unit. — Just as multiples and fractional parts of the real units 1 and -1 are numbers, so we assume that multiples and fractional parts of the new unit i, and of its opposite $-i$, are numbers.

E.g., just as $3 = 1 + 1 + 1$, $-3 = -1 - 1 - 1$,

so $3\sqrt{-1} = \sqrt{-1} + \sqrt{-1} + \sqrt{-1}$, or $3i = i + i + i$;

$-3\sqrt{-1} = -\sqrt{-1} - \sqrt{-1} - \sqrt{-1}$, or $-3i = -i - i - i$;

and $\frac{2}{3}\sqrt{-1} = \frac{\sqrt{-1}}{3} + \frac{\sqrt{-1}}{3}$, or $\frac{2}{3}i = \frac{i}{3} + \frac{i}{3}$.

5. Two or more multiples or fractions of the imaginary unit can be united by addition or subtraction into a single multiple or fraction of that unit.

E.g., $6\sqrt{-1} - 8\sqrt{-1} = -2\sqrt{-1}$, or $6i - 8i = -2i$;

$a\sqrt{-1} + b\sqrt{-1} = (a + b)\sqrt{-1}$, or $ai + bi = (a + b)i$.

6. Multiplication by i. — We define multiplication, when the multiplier is the imaginary unit, by assuming that the Commutative Law holds, that is, by the relation

$$\sqrt{-1} \times a = a\sqrt{-1}, \text{ or } ia = ai.$$

E.g., $i2 = 2i = i + i$.

That is, i is used like a real factor.

7. The following particular cases of Art. 6 deserve special mention:

$$i \cdot 1 = 1 \cdot i = i; \quad i \cdot 0 = 0 \cdot i = 0.$$

8. It follows directly from Arts. 5 and 6 that the Distributive and Associative Laws hold when the imaginary unit is a factor of the product.

E.g., $(a \pm b)i = ai \pm bi$; $aibi = abii = abi^2$.

9. Division by *i*. — It follows from the definition of division that $\frac{ai}{i}$ is a number which multiplied by i gives ai.

But $$a \times i = ai.$$

Therefore $$\frac{ai}{i} = a.$$

Observe again that i is used like a real factor.

10. We now have, in addition to the double series of real numbers, the double series of imaginary numbers:

$$\cdots -3i,\ -2i,\ -i,\ 0,\ i,\ 2i,\ 3i,\ \cdots.$$

Between any two consecutive numbers of this series there are fractional and irrational multiples of i. Thus, between i and $2i$ lie $\frac{3}{2}i$, $\sqrt{2}\,i$, etc.

11. Powers of *i*. — The following values of the positive integral powers of $\sqrt{-1}$, or i, follow directly from the definition of i and Art. 8:

$\sqrt{-1} = \sqrt{-1},$ or $i = i,$

$(\sqrt{-1})^2 = -1,$ $i^2 = -1,$

$(\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = -\sqrt{-1},$ $i^3 = i^2 \cdot i = -i,$

$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = +1,$ $i^4 = i^2 \cdot i^2 = +1,$

$(\sqrt{-1})^5 = (\sqrt{-1})^4(\sqrt{-1}) = +\sqrt{-1},$ $i^5 = i^4 \cdot i = +i,$

$(\sqrt{-1})^6 = (\sqrt{-1})^4(\sqrt{-1})^2 = -1,$ $i^6 = i^4 \cdot i^2 = -1.$

The preceding results give the following properties of powers of i:

(i.) *All even powers of i are real.*

(ii.) *All odd powers of i are imaginary.*

12. Since

$(\sqrt{-a})^2 = -a,$ and $(\sqrt{a} \times \sqrt{-1})^2 = (\sqrt{a})^2(\sqrt{-1})^2 = -a,$

we have $$(\sqrt{-a})^2 = (\sqrt{a} \times \sqrt{-1})^2.$$

Whence $$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}.$$

E.g., $$\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1}.$$

13. Addition of Imaginary Numbers. — Imaginary numbers are united by addition and subtraction just as real numbers are united.

Ex. 1. $\sqrt{-9}+\sqrt{-16}=3\sqrt{-1}+4\sqrt{-1}=7\sqrt{-1}=7\,i.$

Ex. 2. $4\sqrt{-5}-10\sqrt{-5}+3\sqrt{-5}=-3\sqrt{-5}$
$$=-3\sqrt{5}\sqrt{-1}.$$

Ex. 3. $i^{18}+i^{15}=i+(-i)=0.$

14. Multiplication of Imaginary Numbers. — The following principles enable us to simplify a product of imaginary factors:

$$\sqrt{-a}\times\sqrt{b}=\sqrt{(ab)}\times\sqrt{-1}=\sqrt{(-ab)};$$
$$\sqrt{-a}\times\sqrt{-b}=-\sqrt{(ab)}.$$

For
$$\sqrt{-a}\times\sqrt{b}=\sqrt{a}\sqrt{-1}\sqrt{b}=\sqrt{a}\sqrt{b}\sqrt{-1}$$
$$=\sqrt{(ab)}\sqrt{-1}=\sqrt{(-ab)};$$

and
$$\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\sqrt{-1}\times\sqrt{b}\sqrt{-1}=\sqrt{a}\sqrt{b}\,(\sqrt{-1})^2$$
$$=-\sqrt{(ab)}.$$

Ex. 1. $\sqrt{-9}\times\sqrt{16}=3\sqrt{-1}\times 4=12\sqrt{-1}=12\,i.$

Ex. 2. $\sqrt{-2}\times\sqrt{-8}=-\sqrt{16}=-4.$

Ex. 3. $\sqrt{-5}\times\sqrt{-10}\times\sqrt{-15}=\sqrt{5}\times\sqrt{10}\times\sqrt{15}\times(\sqrt{-1})^3$
$$=-5\sqrt{30}\sqrt{-1}=-5\sqrt{30}\cdot i.$$

15. Division of Imaginary Numbers. — The following principles enable us to simplify quotients which involve imaginary numbers:

$$\frac{\sqrt{-a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}\times\sqrt{-1};\quad \frac{\sqrt{a}}{\sqrt{-b}}=-\sqrt{\frac{a}{b}}\times\sqrt{-1};\quad \frac{\sqrt{-a}}{\sqrt{-b}}=\sqrt{\frac{a}{b}}.$$

For
$$\frac{\sqrt{-a}}{\sqrt{b}}=\frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}}=\sqrt{\frac{a}{b}}\times\sqrt{-1}.$$
$$\frac{\sqrt{a}}{\sqrt{-b}}=\frac{\sqrt{a}}{\sqrt{b}\times\sqrt{-1}}=\frac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times(\sqrt{-1})^2}=-\sqrt{\frac{a}{b}}\times\sqrt{-1};$$

and
$$\frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}\sqrt{-1}}=\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}.$$

Ex. 1. $\dfrac{\sqrt{-8}}{\sqrt{2}}=\sqrt{\dfrac{8}{2}}\times\sqrt{-1}=2\sqrt{-1}=2\,i.$

Ex. 2. $\dfrac{1}{\sqrt{-1}}=\dfrac{\sqrt{-1}}{(\sqrt{-1})^2}=\dfrac{\sqrt{-1}}{-1}=-\sqrt{-1}=-\,i.$

EXERCISES I.

Simplify each of the following expressions:

1. $2\sqrt{-9}-3\sqrt{-25}$. **2.** $7\sqrt{-81}+5\sqrt{-144}$.

3. $8\sqrt{-75}+\sqrt{-147}$. **4.** $-5\sqrt{-8}-3\sqrt{-32}$.

5. $2\sqrt{-a^2}+5\sqrt{(-9a^2)}-3\sqrt{(-16a^2)}$.

6. $2\sqrt{(-a^4b)}-4\sqrt{(-a^2b^3)}+2\sqrt{-b^5}$.

7. $\sqrt{3}\cdot\sqrt{-3}$. **8.** $\sqrt{3}\cdot\sqrt{-12}$. **9.** $\sqrt{-50}\cdot\sqrt{-2}$.

10. $\sqrt{-3}\cdot\sqrt{-6}$. **11.** $\sqrt{-2}\cdot\sqrt{-6}\cdot\sqrt{-24}$.

12. $\sqrt{-5}\cdot\sqrt{-20}\cdot\sqrt{8}$. **13.** $\sqrt{6}\cdot\sqrt{-12}\cdot\sqrt{-3}$.

14. $\sqrt{-15}\cdot\sqrt{10}\cdot\sqrt{2}$. **15.** $\sqrt{-x^3}\cdot\sqrt{-x^4}$.

16. $\sqrt{(3x^2)}\cdot\sqrt{-3}$. **17.** $\sqrt{(-a^2b)}\cdot\sqrt{(-ab^3)}\cdot\sqrt{(-ab^2)}$.

18. $\sqrt{(-m^4n^2)}\cdot\sqrt{(-mn^3)}\cdot\sqrt{(-m^3n^7)}\cdot\sqrt{(-m^2n)}$.

19. $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}+\sqrt{-5})$.

20. $(3\sqrt{-5}+4\sqrt{-6})(2\sqrt{-5}-3\sqrt{-6})$.

21. $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$.

22. $[\sqrt{-(a+b)}+\sqrt{-b}][\sqrt{-(a+b)}-\sqrt{-b}]$.

23. $\sqrt{-x^2}$. **24.** $(\sqrt{-x})^2$. **25.** $\sqrt{-a^4}$. **26.** $(\sqrt{-a})^4$.

27. $\sqrt{(1-x)}\cdot\sqrt{(x-1)}$. **28.** $\sqrt{(a-b)}\cdot\sqrt{(b-a)}$. **29.** i^8.

30. i^{18}. **31.** i^{14}. **32.** i^{101}. **33.** i^4+i^{34}. **34.** $i^{35}-i^{91}$.

35. $(\sqrt{-a})^{29}$. **36.** $(-a\sqrt{-a})^{42}$. **37.** $\sqrt{-27}\div\sqrt{-3}$.

38. $\sqrt{-8}\div\sqrt{-2}$. **39.** $5\sqrt{-35}\div 2\sqrt{7}$. **40.** $\sqrt{3}\div\frac{1}{2}\sqrt{-5}$.

41. $\sqrt{-a}\div\sqrt{-a^2}$. **42.** $\sqrt{(-ab)}\div\sqrt{-b}$.

43. $(\sqrt{-6}+\sqrt{-8})\div\sqrt{-2}$. **44.** $(\sqrt{-12}-\sqrt{18})+\sqrt{-3}$.

45. $\frac{1}{i}$. **46.** $\frac{1}{i^7}$. **47.** $\frac{1}{i^6}$. **48.** $\frac{1}{i+i^5}$. **49.** $\frac{1}{i^4+i^8}$.

Complex Numbers.

16. A **Complex Number** is the algebraic sum of a real and an imaginary number; as, $3\pm 2i$.

The general form of a complex number is evidently $a+bi$, wherein a and b are real numbers.

When $b=0$, we have any real number.

When $a=0$, we have any imaginary number.

17. Two complex numbers which differ only in the sign of their imaginary terms are called **Conjugate Complex Numbers**; as, $2-3i$ and $2+3i$.

18. *Two complex numbers are said to be equal when the real term of one is equal to the real term of the other, and the imaginary term of one is equal to the imaginary term of the other;* as, $2+3i=2+3i$.

That is, if $$a+bi=c+di,$$
then $$a=c, \text{ and } bi=di, \text{ or } b=d.$$

Observe that the preceding statement is a definition of the meaning of the sign of equality between two complex numbers.

19. From the preceding article it follows that, if

$$a+bi=0=0+0i, \text{ then } a=0,\ b=0.$$

20. Addition and Subtraction of Complex Numbers. — The following definition of Addition and Subtraction of Complex Numbers is a natural extension of the definition of these operations for real numbers:

Two complex numbers are added or subtracted by adding or subtracting the real parts by themselves and the imaginary parts by themselves.

E.g., $(2+3i)+(-5+6i)=(2-5)+(3i+6i)=-3+9i$.

In general, $(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i$.

21. The Commutative and Associative Laws hold for algebraic addition of complex numbers.

This principle follows immediately from the definition of addition and subtraction.

That is, $(a+bi)+(c+di)=(c+di)+(a+bi)$;

$$(a+bi)+(c+di)+(e+fi)=(a+bi)+[(e+fi)+(c+di)].$$

22. The sum or difference of two complex numbers is, in general, a complex number.

E.g., $(2+3i)+(-4+2i)=(2-4)+(3+2)i=-2+5i$.

But the sum of two conjugate complex numbers is real.

E.g., $$(2+3i)+(2-3i)=4.$$

23. Multiplication of Complex Numbers. — We define multiplication by a complex number by assuming that the Distributive Law holds; that is, by the relation

$$(a+bi)(c+di) = ac + bci + adi + bdi^2,$$

or $$(a+bi)(c+di) = (ac-bd) + (bc+ad)i.$$

24. The Commutative, Associative, and Distributive Laws hold for multiplication of complex numbers.

This principle follows from the definition of multiplication.

That is, $(a+bi)(c+di) = (c+di)(a+bi)$;

$$(a+bi)(c+di)(e+fi) = (a+bi)[(e+fi)(c+di)] = \text{etc.}$$

25. The product of two complex numbers is, in general, a complex number.

E.g., $(2+3i)(-4+2i) = -8-12i+4i-6 = -14-8i.$

But the product of two conjugate complex numbers is *real and positive.*

E.g., $(-2+3i)(-2-3i) = (-2)^2-(3i)^2 = 4+9 = 13.$

In general, $(a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2.$

26. The square of a complex number is a complex number.

E.g., $$\begin{aligned}(2+3\sqrt{-1})^2 &= 4+12\sqrt{-1}+(3\sqrt{-1})^2\\ &= 4+12\sqrt{-1}-9\\ &= -5+12\sqrt{-1} = -5+12i.\end{aligned}$$

But the cube of a complex number is sometimes real.

E.g., $(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})^3 = -\frac{1}{8} \pm \frac{3}{8}\sqrt{-3} + \frac{9}{8} \mp \frac{3}{8}\sqrt{-3} = 1.$

From these results we see that $\sqrt[3]{1}$ has three values,

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

27. Division of Complex Numbers. — The quotient of one complex number by another is in general a complex number.

For $$\begin{aligned}\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\cdot i.\end{aligned}$$

28. It follows from the preceding article that a fraction whose denominator is a complex number can be expressed as a complex number by multiplying both numerator and denominator by the conjugate of the denominator.

Ex. 1. $$\frac{1+\sqrt{-2}}{2\sqrt{-3}}=\frac{(1+\sqrt{-2})(-\sqrt{-3})}{(2\sqrt{-3})(-\sqrt{-3})}=\frac{-\sqrt{-3}+\sqrt{6}}{6}$$
$$=\tfrac{1}{6}\sqrt{6}-\tfrac{1}{6}\sqrt{-3}.$$

Notice that it was necessary to multiply numerator and denominator only by $-\sqrt{-3}$.

Ex. 2.

$$\frac{1}{2+\sqrt{-3}}=\frac{2-\sqrt{-3}}{(2+\sqrt{-3})(2-\sqrt{-3})}=\frac{2-\sqrt{-3}}{7}=\tfrac{2}{7}-\tfrac{1}{7}\sqrt{-3}.$$

Complex Factors.

29. A quadratic expression which is the product of two complex factors can be resolved into factors by the method used to resolve a quadratic expression into irrational factors.

Ex. Factor x^2-2x+3.

Completing x^2-2x to the square of a binomial in x, we have

$$\begin{aligned}x^2-2x+3&=x^2-2x+1-1+3\\&=(x-1)^2-(\sqrt{-2})^2\\&=(x-1+\sqrt{-2})(x-1-\sqrt{-2}).\end{aligned}$$

Square Root of a Complex Number.

30. If $\sqrt{(a+bi)}=\sqrt{x}+i\sqrt{y},$

then $\sqrt{(a-bi)}=\sqrt{x}-i\sqrt{y}.$

For, from $\sqrt{(a+bi)}=\sqrt{x}+i\sqrt{y},$

we have $a+bi=x-y+2\sqrt{(xy)}\cdot i.$

Therefore, by Art. 18, $a=x-y$, and $b=2\sqrt{(xy)}$.

Consequently, $$\begin{aligned}a-bi&=x-y-2\sqrt{(xy)}\cdot i\\&=(\sqrt{x}-i\sqrt{y})^2.\end{aligned}$$

Whence $\sqrt{(a-bi)}=\sqrt{x}-i\sqrt{y}.$

31. The square root of a complex number can be expressed as a complex number.

Assuming $\sqrt{(13-20\sqrt{-3})}=\sqrt{x}-i\sqrt{y},$ (1)

we have $\sqrt{(13+20\sqrt{-3})}=\sqrt{x}+i\sqrt{y}.$ (2)

Multiplying (1) by (2),

$$\sqrt{(169+1200)}=x+y,$$

or $x+y=37.$ (3)

Squaring (1), $13-20\sqrt{-3}=x-y-2\sqrt{(xy)}\cdot i;$

whence $x-y=13.$ (4)

From (3) and (4), $x=25,\ y=12.$

Therefore $\sqrt{(13-20\sqrt{-3})}=5-2\sqrt{3}\cdot i.$

In general, assuming $\sqrt{(a+bi)}=\sqrt{x}+i\sqrt{y},$ (1)

we have $\sqrt{(a-bi)}=\sqrt{x}-i\sqrt{y}.$ (2)

Multiplying (1) by (2), $\sqrt{(a^2+b^2)}=x+y.$ (3)

Squaring (1), $a+bi=x-y+2\sqrt{(xy)}\cdot i;$

whence $a=x-y.$ (4)

From (3) and (4), $x=\dfrac{\sqrt{(a^2+b^2)}+a}{2},\ y=\dfrac{\sqrt{(a^2+b^2)}-a}{2}.$

Therefore $\sqrt{(a+bi)}=\sqrt{\dfrac{\sqrt{(a^2+b^2)}+a}{2}}+i\sqrt{\dfrac{\sqrt{(a^2+b^2)}-a}{2}}.$

Since a and b are real, therefore $\sqrt{(a^2+b^2)}$ is real.

32. Assuming $a=0$, in the general result of the preceding article, we have

$$\sqrt{(bi)}=\sqrt{\frac{b}{2}}+i\sqrt{\frac{b}{2}}=\tfrac{1}{2}\sqrt{(2b)}\cdot(1+i).$$

In particular, if $b=1$,

$$\sqrt{i}=\sqrt[4]{-1}=\tfrac{1}{2}\sqrt{2}\cdot(1+i).$$

EXERCISES II.

Simplify each of the following expressions:

1. $(1+\sqrt{-9})+(4-\sqrt{-4}).$
2. $(6-\sqrt{-16})-(-5-\sqrt{-36}).$
3. $(2+4i)+(-3+2i).$
4. $(7-5i)-(3-4i).$
5. $(\sqrt{\tfrac{1}{2}}-\tfrac{1}{2}\sqrt{-2})\sqrt{-8}.$
6. $(\sqrt{5}-\sqrt{-3})\sqrt{-3}.$
7. $(2+3\sqrt{-1})(3-4\sqrt{-1}).$
8. $(5+3\sqrt{-1})(5-3\sqrt{-1}).$
9. $(5-2\sqrt{-6})(3-4\sqrt{-3}).$
10. $(7+\sqrt{-5})(7-\sqrt{-5}).$

11. $(\sqrt{12} - 3i)(\sqrt{3} + 5i)$. **12.** $(2 + i\sqrt{3})(2 - i\sqrt{3})$.

13. $(2\sqrt{3}+5\sqrt{-7})(2\sqrt{3}-5\sqrt{-7})$. **14.** $(\sqrt{8} - \sqrt{-12})(\sqrt{2} + \sqrt{-3})$.

15. $(\frac{1}{4} - \frac{1}{4}\sqrt{3} \cdot i)(3 + 3\sqrt{3} \cdot i)$. **16.** $(\sqrt{5} - 2i\sqrt{6})(\sqrt{5} + 2i\sqrt{6})$.

17. $[a - b + \sqrt{(-2ab)}][a - b - \sqrt{(-2ab)}]$.

18. $[x+i\sqrt{(a-x^2)}][x-i\sqrt{(a-x^2)}]$. **19.** $\sqrt{(1+\sqrt{-1})} \times \sqrt{(1-\sqrt{-1})}$.

20. $\sqrt{(\sqrt{2} + \sqrt{3} \cdot i)} \times \sqrt{(\sqrt{2} - \sqrt{3} \cdot i)}$.

21. $(1 + \sqrt{-2})^2$. **22.** $(\sqrt{-2} + \sqrt{3})^2$. **23.** $(\frac{1}{2}\sqrt{2} \pm \frac{1}{3}\sqrt{-3})^3$.

24. $(3 - 5\sqrt{-2})^3$. **25.** $[\frac{1}{3}\sqrt{3}(3 - i)]^3$. **26.** $(\sqrt{-75} - 3)^4$.

27. $(1 + 2ai)^2$. **28.** $(a + bi)^3$. **29.** $(\sqrt{a} + \sqrt{b} \cdot i)^4$.

Reduce each of the following expressions to the form of a complex number:

30. $\dfrac{6\sqrt{3} - \sqrt{-15}}{-2\sqrt{-3}}$. **31.** $\dfrac{8i + 6\sqrt{2} \cdot i + 2}{4i}$.

32. $\dfrac{3}{1 + \sqrt{-2}}$. **33.** $\dfrac{7}{2 - \sqrt{-3}}$. **34.** $\dfrac{14}{3 + 2\sqrt{5} \cdot i}$.

35. $\dfrac{3 + 2\sqrt{-1}}{2 - 3\sqrt{-1}}$. **36.** $\dfrac{3 + 4\sqrt{-5}}{4 - 3\sqrt{-5}}$. **37.** $\dfrac{1 + i}{(1 + i)^3}$.

38. $\dfrac{2}{\sqrt{2} - \sqrt{-2} - 2\sqrt{-1}}$. **39.** $\dfrac{10 + \sqrt{-5}}{1 - \sqrt{-3} + \sqrt{-5}}$.

Factor each of the following expressions:

40. $x^2 - 6x + 29$. **41.** $x^2 + 4x + 67$. **42.** $x^2 - 14x + 61$.

43. $5x^2 - 6x + 2$. **44.** $4x^2 + 4xy + 3y^2$. **45.** $16x^2 - 8xy + 5y^2$.

Find the square root of each of the following expressions:

46. $1 + \sqrt{-3}$. **47.** $5 - \sqrt{-11}$. **48.** $3 + 4i$.

49. $-3 + 4i$. **50.** $-15 + 3\sqrt{-11}$. **51.** $2a + a\sqrt{-5}$.

Make the indicated substitution in each of the following expressions, and simplify the results:

52. In $x^2 - 6x + 14$, let $x = 3 + \sqrt{-5}$.

53. In $3x^2 - 5x + 7$, let $x = 2 - 3\sqrt{-2}$.

54. In $5x^3 + 2x^2 - 3x - 1$, let $x = 1 - 2i$.

Simplify each of the following expressions:

55. $\dfrac{5}{4 - \sqrt{-14}} - \dfrac{3}{2 - \sqrt{-14}}$. **56.** $\dfrac{\sqrt{-2}}{6 + \sqrt{-6}} + \dfrac{\sqrt{-\frac{1}{2}}}{3 - \sqrt{-\frac{3}{2}}}$.

57. $\dfrac{3}{1 + i} - \dfrac{5}{4 - 2i} + \dfrac{4}{1 - i}$. **58.** $\dfrac{a + bi}{a - bi} \pm \dfrac{a - bi}{a + bi}$.

CHAPTER XXI.

QUADRATIC EQUATIONS.

1. A **Quadratic Equation** is an equation of the second degree in the unknown number or numbers.

E.g., $x^2 = 25$, $x^2 - 5x + 6 = 0$, $x^2 + 2xy = 7$.

A **Complete Quadratic Equation**, in one unknown number, is one which contains a term (or terms) in x^2, a term (or terms) in x, and a term (or terms) free from x, as $x^2 - 2ax + b = cx - d$.

A **Pure Quadratic Equation** is an incomplete quadratic equation which has no term in x, as $x^2 - 9 = 0$.

In this chapter we shall consider quadratic equations in only one unknown number.

2. The following example illustrates a principle of the equivalence of a quadratic equation to two derived linear equations.

The equation $x^2 + 6x + 9 = 16$, or $(x + 3)^2 = 16$, (1)

is equivalent to the two equations

$$x + 3 = 4, \text{ and } x + 3 = -4, \tag{2}$$

obtained by equating the positive square root of the first member in turn to the positive and to the negative square root of the second member.

For (1) is equivalent to $(x + 3)^2 - 16 = 0$. (3)

This equation is equivalent to

$$x + 3 - 4 = 0 \text{ and } x + 3 + 4 = 0$$

jointly. But the latter equations are equivalent to

$$x + 3 = 4,\ x + 3 = -4.$$

Equations (2) are usually written $x + 3 = \pm 4$.

In general, *if the positive square root of the first member of an equation be equated in turn to the positive and to the negative square root of the second member, these two derived equations are jointly equivalent to the given equation.*

For, the equation $M = N$

is equivalent to $M - N = 0;$

that is, to $(\sqrt{M} + \sqrt{N})(\sqrt{M} - \sqrt{N}) = 0.$

The last equation is equivalent to

$$\sqrt{M} - \sqrt{N} = 0 \text{ and } \sqrt{M} + \sqrt{N} = 0$$

jointly; that is, to $\sqrt{M} = \pm\sqrt{N}.$

Pure Quadratic Equations.

3. Any pure quadratic equation can be reduced to the form $x^2 = m$. From this equation we obtain $x = \pm\sqrt{m}$, by Art. 2.

Ex. Solve the equation $(2x - 5)(2x + 5) = 11.$

Simplifying, $x^2 = 9$; whence $x = \pm 3.$

EXERCISES I.

Solve each of the following equations:

1. $x^2 = 289.$ **2.** $x^2 = 2809.$ **3.** $x^2 = 3.61.$

4. $x^2 = 53.29.$ **5.** $\frac{2}{3}x^2 = 1536.$ **6.** $\frac{4}{5}x^2 = 1479.2.$

7. $9x^2 - 36 = 5x^2.$ **8.** $7x^2 - 8 = 9x^2 - 10.$

9. $(3x - 4)(3x + 4) = 65.$ **10.** $(7 + x)^2 + (7 - x)^2 = 130.$

11. $(2x - 3)(3x - 4) - (x - 13)(x - 4) = 40.$

12. $(5x - 7)(3x + 8) - (x - 10)(9 - x) = 1634.$

13. $(4 + x)(3 - x)(2 - x) - (x + 2)(x + 3)(x - 4) = 30.$

14. $(5 - x)(3 - x)(1 + x) + (5 + x)(3 + x)(1 - x) = 16.$

15. $8(2 - x)^2 = 2(8 - x)^2.$ **16.** $(3 - x)^2 = 3(1 - x)^2.$

17. $ax^2 + b = bx^2 + a.$ **18.** $a(x^2 + b) = b(x^2 + a).$

Solution by Factoring.

4. The principle on which the solution of an equation by factoring depends was proved in Ch. VIII., § 4, Art. 1. The methods given in Ch. VIII., § 1, Arts. 9–13; Ch. XIX., Art. 20, and Ch. XX., Art. 29, enable us to factor any quadratic expression. The roots of the given quadratic equation are the roots of the equations obtained by equating to 0 each of its factors.

Ex. 1. Solve the equation $4(x - \frac{3}{2})^2 = 6x + 20$.

Reducing the first member,

$$4x^2 - 12x + 9 = 6x + 20.$$

Transferring and uniting terms,

$$4x^2 - 18x - 11 = 0.$$

Factoring first member,

$$4(x - \tfrac{9}{4} + \tfrac{5}{4}\sqrt{5})(x - \tfrac{9}{4} - \tfrac{5}{4}\sqrt{5}) = 0.$$

Equating each factor to 0, $x - \frac{9}{4} + \frac{5}{4}\sqrt{5} = 0$,

$$x - \tfrac{9}{4} - \tfrac{5}{4}\sqrt{5} = 0.$$

Whence $x = \frac{9}{4} - \frac{5}{4}\sqrt{5}$, and $x = \frac{9}{4} + \frac{5}{4}\sqrt{5}$.

Ex. 2. Solve the equation $4m^2x^2 + 4m^2n + 1 = 4mx$.

Transferring terms, $4m^2x^2 - 4mx + 1 + 4m^2n = 0$,

or $(2mx - 1)^2 - (2m\sqrt{-n})^2 = 0$.

Equating to 0 the factors of the first member,

$$2mx - 1 + 2m\sqrt{-n} = 0,$$

$$2mx - 1 - 2m\sqrt{-n} = 0.$$

Whence $x = \dfrac{1}{2m} - \sqrt{-n}$, and $x = \dfrac{1}{2m} + \sqrt{-n}$.

EXERCISES II.

Solve each of the following equations:

1. $x^2 - 7x = 4x$.
2. $x^2 - 2x - 17 = 0$.
3. $x^2 - 6x + 8 = 0$.
4. $x^2 - 4x + 8 = 0$.
5. $x^2 - 4x - 71 = 0$.
6. $x^2 + 10x + 24 = 0$.
7. $13x - 6 - 6x^2 = 0$.
8. $(x + 10)^2 = 28$.
9. $6x - x^2 = 18$.
10. $7x^2 - 3x = 160$.
11. $(2x - 1)^2 = 2$.
12. $x(5x - 2) = -6$.
13. $x^2 - 2\sqrt{2}x - 1 = 0$.
14. $36x^2 - 36\sqrt{5}x + 17 = 0$.
15. $(x + 6)(x + 3) = x - 6$.
16. $(x + 7)(x - 7) = 2(x + 50)$.
17. $(2x + 1)(x + 2) = 3x^2 - 4$.
18. $(x - 1)(2x + 3) = 4x^2 - 22$.
19. $x^2 - 3 = \frac{1}{6}(x - 3)$.
20. $x(x + 5) = 5(40 - x) + 27$.

21. $7x(x-1)=7-4(x-1)$.
22. $(2x+1)^2=x(x+2)$.
23. $x^2-2ax+a^2=b^2$.
24. $x^2-2mx-1=0$.
25. $x^2-2ax+a^2+b^2=0$.
26. $x^2-4b^2=a(2x-5a)$.
27. $n^2x^2+2mnx+2m^2=0$.
28. $x^2-(a+b)x+ab=0$.
29. $(a^2+b^2)x-abx^2-ab=0$.
30. $x^2+2mx+m^2=n$.
31. $x^2+2a+1=2(x-a)$.
32. $a^2x^2-2ax+1=a^2$.
33. $4x^2-12ax+9a^2=4b^2$.
34. $(x+a)^2=5ax-(x-a)^2$.
35. $x^2-4(a+b)+1=2x$.
36. $x^2=2(a+b)x+2(a^2+b^2)$.
37. $(m^2-1)x^2-2(m^2+1)x+m^2-1=0$.
38. $mnx^2-(m+n)(mn+1)x+(m^2+1)(n^2+1)=0$.
39. $(m-n)^2(m+n)x^2+2(m-n)(m+n)^2x+6m^2n+2n^3=0$.

Solution by Completing the Square.

5. The following examples illustrate the solution of a quadratic equation by the method called *Completing the Square*.

Ex. 1. Solve the equation $x^2-5x+6=0$.

Transferring 6, $\qquad x^2-5x=-6$.

To complete the square in the first member, we add $(-\frac{5}{2})^2$, $=\frac{25}{4}$, to this member, and therefore also to the second. We then have

$$x^2-5x+\tfrac{25}{4}=\tfrac{25}{4}-6=\tfrac{1}{4}.$$

Equating square roots, $x-\frac{5}{2}=\pm\frac{1}{2}$, by Art. 2.

Whence $\qquad x=\frac{5}{2}\pm\frac{1}{2}$.

Therefore the required roots are 3 and 2.

Ex. 2. Solve the equation

$$7x^2+5x+1=0.$$

Transferring 1, $\qquad 7x^2+5x=-1$.

Dividing by 7, $\qquad x^2+\frac{5}{7}x=-\frac{1}{7}$.

Adding $(\frac{5}{2\times 7})^2=\frac{25}{196}$, $x^2+\frac{5}{7}x+\frac{25}{196}=\frac{25}{196}-\frac{1}{7}=\frac{-3}{196}$.

Equating square roots, $x+\frac{5}{14}=\pm\frac{\sqrt{-3}}{14}$.

Whence $\qquad x=-\frac{5}{14}\pm\frac{\sqrt{-3}}{14}$.

Therefore the required roots are

$$-\tfrac{5}{14}+\tfrac{1}{14}\sqrt{-3} \text{ and } -\tfrac{5}{14}-\tfrac{1}{14}\sqrt{-3}.$$

Ex. 3. Solve the equation

$$(a^2 - b^2)x^2 - 2a^2x + a^2 = 0.$$

Transferring a^2, $\quad (a^2 - b^2)x^2 - 2a^2x = -a^2.$

Dividing by $a^2 - b^2$, $\quad x^2 - \dfrac{2a^2x}{a^2 - b^2} = \dfrac{-a^2}{a^2 - b^2}.$

Adding $\left(-\dfrac{a^2}{a^2 - b^2}\right)^2, = \dfrac{a^4}{(a^2 - b^2)^2}$, to both members,

$$x^2 - \frac{2a^2x}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = -\frac{a^2}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = \frac{a^2b^2}{(a^2 - b^2)^2}.$$

Equating square roots, $x - \dfrac{a^2}{a^2 - b^2} = \pm \dfrac{ab}{a^2 - b^2}.$

Whence $\quad x = \dfrac{a^2 \pm ab}{a^2 - b^2}.$

Therefore the required roots are $\dfrac{a}{a - b}$ and $\dfrac{a}{a + b}$.

The preceding examples illustrate the following method of procedure:

Bring the terms in x and x^2 to the first member, and the terms free from x to the second member, uniting like terms.

If the resulting coefficient of x^2 be not $+1$, divide both members by this coefficient.

Complete the square by adding to both members the square of half the coefficient of x.

Equate the positive square root of the first member to the positive and negative square roots of the second member.

Solve the resulting equations.

EXERCISES III.

Solve each of the following equations:

1. $x^2 - 4x + 3 = 0.$
2. $x^2 - 5x = -4.$
3. $x^2 + 2x + 1 = 0.$
4. $2x^2 - 7x + 3 = 0.$
5. $3x^2 - 53x + 34 = 0.$
6. $14x - 49x^2 - 1 = 0.$
7. $x^2 - 4x + 7 = 0.$
8. $(2x - 1)(x - 2) = (x + 1)^2.$
9. $x^2 - 2x + 6 = 0.$
10. $x^2 - 1 + x(x - 1) = x^2.$
11. $(3x - 2)(x - 1) = 14.$
12. $110x^2 - 21x + 1 = 0.$

13. $(2x-3)^2=8x.$ **14.** $(5x-3)^2-7=40x-47.$

15. $(2x+1)(x+2)=3x^2-4.$ **16.** $(x+1)(2x+3)=4x^2-22.$

17. $(x-7)(x-4)+(2x-3)(x-5)=103.$

18. $10(2x+3)(x-3)+(7x+3)^2=20(x+3)(x-1).$

19. $(x-1)(x-3)+(x-3)(x-5)=32.$

20. $(x-1)(x-2)+(x-3)(x-4)=(x-1)^2-2.$

21. $(m-n)x^2-(m+n)x+2n=0.$ **22.** $4x^2-4(3a+2b)x+24ab=0.$

23. $(a+b)^2x^2+2abx=-\dfrac{a^2b^2}{(a+b)^2}.$

24. $x^2-2(a+b)x+(a+b+c)(a+b-c)=0.$

25. $x^2-(a-1)x-a=0.$ **26.** $x^2-2cx+ac+bc-ab=0.$

27. $x^2-4mnx=(m^2-n^2)^2.$ **28.** $d^2x^2-4abdx+4a^2b^2-9c^2=0.$

29. $(a^2-b^2)x^2-2(a^2+b^2)x+a^2-b^2=0.$

30. $abcx^2-(a^2b^2+c^2)x+abc=0.$ **31.** $x^2\sqrt{6}-(\sqrt{2}+\sqrt{3})x+1=0.$

32. $x^2-2x\sqrt{(a+b)}+2b=0.$ **33.** $\dfrac{(x-4a)(x+2b-2)}{(a+b-1)(a-b+1)}=-3.$

34. $\dfrac{x^2}{a^2+ab+ac}=\dfrac{b+c-x}{a+b+c}-\dfrac{(b+c)x^2}{a^3+a^2b+a^2c}.$

General Solution.

6. The most general form of the quadratic equation in one unknown number is evidently

$$ax^2+bx+c=0.$$

The coefficient a is assumed to be *positive* and not 0, but b and c may either or both be positive or negative, or 0.

Dividing by a, $$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Transferring $\dfrac{c}{a}$, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Adding $\left(\dfrac{b}{2a}\right)^2,=\dfrac{b^2}{4a^2}$, $$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$
$$=\frac{b^2-4ac}{4a^2}.$$

Equating square roots, $$x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}.$$

Whence $$x = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

and $$x = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}.$$

7. The roots of any quadratic equation can be obtained by substituting in the general solution the particular values of the coefficients a, b, and c.

Ex. Solve the equation $3x^2 + 7x - 10 = 0$.

We have $$a = 3,\ b = 7,\ c = -10.$$

Substituting these values in the general solution, we obtain

$$x = -\tfrac{7}{6} + \frac{\sqrt{[49 - 4 \times 3(-10)]}}{6} = 1,$$

and $$x = -\tfrac{7}{6} - \frac{\sqrt{[49 - 4 \times 3(-10)]}}{6} = -\tfrac{10}{6}.$$

EXERCISES IV.

Solve each of the following equations:

1. $2x^2 = 3x + 2$.
2. $5x^2 - 6x + 1 = 0$.
3. $9x(x + 1) = 28$.
4. $x^2 - b^2 = 2ax - a^2$.
5. $x^2 + 6ax + 1 = 0$.
6. $x^2 + 1 = 2\tfrac{1}{6}x$.
7. $(x - 5)^2 + (x - 10)^2 = 37$.
8. $2x(3n - 4x) = n^2$.
9. $n^2(x^2 + 1) = a^2 + 2n^2x$.
10. $x^2 + (x + a)^2 = a^2$.

Fractional Equations which lead to Quadratic Equations.

8. The principles given in Ch. X. for solving fractional equations which lead to linear equations hold also for fractional equations which lead to quadratic equations.

Ex. 1. Solve the equation $$\frac{4}{x - 1} = \frac{3x}{x^2 - 1} + 2. \qquad (1)$$

Multiplying by $x^2 - 1$, $$4(x + 1) = 3x + 2(x^2 - 1). \qquad (2)$$

Transferring and uniting terms, $$2x^2 - x = 6. \qquad (3)$$

Dividing by 2, $$x^2 - \tfrac{1}{2}x = 3. \qquad (4)$$

The roots of equation (4) are 2, $-\tfrac{3}{2}$. Since neither is a root of the L. C. D. (equated to 0) of the fractions in the given equation, *i.e.*, of $x^2 - 1 = 0$, they are the roots of that equation.

Ex. 2. Solve the equation

$$\frac{1}{n+x}-\frac{1}{n-x}=\frac{x^2-2n-n^2}{x^2-n^2}. \qquad (1)$$

Uniting the fractions in the first member,

$$\frac{-2x}{n^2-x^2}=\frac{x^2-2n-n^2}{x^2-n^2}. \qquad (2)$$

Clearing of fractions,

$$2x=x^2-2n-n^2. \qquad (3)$$

Transferring and uniting terms,

$$x^2-2x=n^2+2n. \qquad (4)$$

Completing the square,

$$x^2-2x+1=n^2+2n+1. \qquad (5)$$

Equating square roots,

$$x-1=\pm(n+1).$$

Therefore the roots of (5) are

$$1+(n+1)=2+n, \text{ and } 1-(n+1)=-n.$$

The number $2+n$ is not a root of the L. C. D. equated to 0, that is, of $$x^2-n^2=0.$$

Therefore $2+n$ is a root of the given equation.

But $-n$ is a root of $x^2-n^2=0$, or of $(x-n)(x+n)=0$, and is therefore not a root of the given equation.

This root was introduced by multiplying the given equation by the factor $x+n$ which was not necessary to clear it of fractions.

For, transferring and uniting the fractions in equation (2),

we obtain $$\frac{x^2-2x-2n-n^2}{x^2-n^2}=0.$$

Factoring the numerator,

$$\frac{(x+n)(x-n-2)}{x^2-n^2}=0.$$

Reducing to lowest terms, $$\frac{x-n-2}{x-n}=0.$$

The numerator equated to 0 gives $x-n-2=0$, whence $x=2+n$, as above.

9. The work of solving an equation can sometimes be simplified by a simple substitution.

Ex. Solve the equation $\frac{x+5}{x+2} - \frac{x+2}{x+5} = \frac{3}{2}$.

If we let $\frac{x+5}{x+2} = y$, the given equation becomes $y - \frac{1}{y} = \frac{3}{2}$.

The roots of this equation are 2, $-\frac{1}{2}$.

We now have to solve the two equations

$$\frac{x+5}{x+2} = 2, \text{ whence } x = 1;$$

and

$$\frac{x+5}{x+2} = -\tfrac{1}{2}, \text{ whence } x = -4.$$

This method can be used when the fractional equation contains only two expressions in the unknown number, one of which is the reciprocal of the other.

EXERCISES V.

Solve each of the following equations:

1. $15x + \frac{2}{x} = 11$.
2. $x + \frac{1}{x} = 7 + \frac{1}{7}$.
3. $\frac{x-6}{x+30} = \frac{3}{x-6}$.
4. $\frac{x}{x+120} = \frac{14}{3x-10}$.
5. $\frac{x}{x-6} + 1 = \frac{60}{x+4}$.
6. $\frac{3}{1+x} + \frac{3}{1-x} = 8$.
7. $\frac{x-1}{x} - \frac{3x}{x-1} = 2$.
8. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}$.
9. $\frac{x+1}{x-1} + \frac{x+3}{x-3} = \frac{11}{2}$.
10. $\frac{7}{x} - \frac{15}{x+2} + \frac{5}{x-8} = 0$.
11. $\frac{3}{2(x^2-1)} - \frac{1}{4(x+1)} = \frac{1}{8}$.
12. $\frac{9x+1}{9x-3x^2} = \frac{x}{21-7x} - \frac{x+3}{21x}$.
13. $\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} = 0$.
14. $\frac{5x-1}{x+3} + \frac{7x^2-106}{8x^2-72} = -\frac{1}{8}$.
15. $\frac{x-2}{x+2} + \frac{x+2}{x-2} = \frac{2(x+3)}{x-3}$.
16. $\frac{x+24}{5x^2-5} = \frac{x-7}{x+1} - \frac{1}{2x-2}$.
17. $\frac{4x+67}{40x^2-36} + \frac{x}{30x^2-27} = \frac{2}{3}$.
18. $\frac{x^3+3x^2+3x+1}{x^3+2x^2+2x+1} = \frac{9}{7}$.
19. $\frac{3x-16}{2x-12} + \frac{2x-12}{3x-16} = \frac{5}{2}$.
20. $\frac{2x-2}{5x+5} = \frac{x^2-1}{x^2+1}$.
21. $x - \frac{a}{b} = \frac{b}{a} - \frac{1}{x}$.
22. $\frac{m(m-1)}{x-1} = x+1$.
23. $\frac{x-1}{a} = \frac{b}{x+1}$.
24. $\frac{x-a}{x+a} = \frac{b-x}{b+x}$.
25. $\frac{x}{x-1} - \frac{x}{x+1} = m$.

26. $\dfrac{an}{x+4n}-\dfrac{an}{x-4n}=2.$

27. $x=\dfrac{3}{(a-b)^2x}-\dfrac{2}{a-b}.$

28. $\dfrac{1}{(a^2-x)^2}=\dfrac{4x}{4x^3+5a^6}.$

29. $\dfrac{x^2+1}{n^2x-2n}-\dfrac{1}{2-nx}=\dfrac{x}{n}.$

30. $\dfrac{a-2b}{8x^2-2b^2}=\dfrac{1}{2x+b}-\dfrac{1}{2a}.$

31. $1-\dfrac{2a}{x-a}=\dfrac{a^2+b^2}{a^2+x^2-2ax}.$

32. $\dfrac{a}{nx-x}-\dfrac{a-1}{x^2-2nx^2+n^2x^2}=1.$

33. $\dfrac{n+x}{n-x}+\dfrac{n-x}{n+x}=\dfrac{n^2}{n^2-x^2}.$

34. $\dfrac{x-a+b}{x+a-b}=\dfrac{a-b-x}{a+b+x}.$

35. $\dfrac{ax}{ax+1}=\dfrac{1-a}{a^2x^2-a-a^2x+ax}.$

36. $\left(\dfrac{a+x}{a-x}\right)^2+\dfrac{7}{2}\cdot\dfrac{a+x}{a-x}+3=0.$

37. $\dfrac{11a-x}{3a-x}=\dfrac{121a^2-x^2}{9a^2-x^2}.$

38. $\dfrac{x+1}{x-1}+\dfrac{a-b}{a+b}=\dfrac{x-1}{x+1}+\dfrac{a+b}{a-b}.$

39. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}.$

40. $\dfrac{x-b}{x-a}+\dfrac{x-a}{x-b}=\dfrac{a}{b}+\dfrac{b}{a}.$

41. $\dfrac{(a-x)^3+(x-b)^3}{(a-x)^2+(x-b)^2}=a-b.$

42. $\dfrac{(a-x)^3+(x-b)^3}{(a-x)^2+(x-b)^2}=\dfrac{a^3-b^3}{a^2+b^2}.$

43. $\dfrac{(a-x)^3+(x-b)^3}{(a+b-2x)^2}=\dfrac{a^3-b^3}{(a+b)^2}.$

44. $\dfrac{x^2-2nx+2ax-n^2}{x^3-a^3}+\dfrac{x+2n}{x^2+ax+a^2}=\dfrac{1}{x-a}.$

Theory of Quadratic Equations.

10. *A quadratic equation has two, and only two, roots.*

For, by Art. 6, the equations

$$x=\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a}$$

and

$$x=\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}$$

are jointly equivalent to the equation $ax^2+bx+c=0$.

Therefore ax^2+bx+c has two, and only two, roots.

11. Relations between the roots of a quadratic equation and the coefficients of its terms.—If the roots of the quadratic equation

$$ax^2+bx+c=0, \text{ or } x^2+\frac{b}{a}x+\frac{c}{a}=0$$

be designated by r_1 and r_2, we have

$$r_1=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a},$$

$$r_2=-\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$$

The sum of the roots is $r_1 + r_2 = -\frac{b}{a}$. (1)

The product of the roots is

$$r_1r_2 = \left[-\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}\right] \times \left[-\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}\right]$$

$$= \left[-\frac{b}{2a}\right]^2 - \left[\frac{\sqrt{(b^2 - 4ac)}}{2a}\right]^2 = \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{c}{a}. \quad (2)$$

The relations (1) and (2) may be expressed thus:

(i.) *The sum of the roots of a quadratic equation is equal to the quotient obtained by dividing the coefficient of the first power of the unknown number, with sign reversed, by the coefficient of the second power of the unknown number.*

In particular, if the coefficient of the second power of the unknown number be 1, *the sum of the roots is equal to the coefficient of the first power of the unknown number, with sign reversed.*

(ii.) *The product of the roots of a quadratic equation is equal to the quotient obtained by dividing the term free from the unknown number by the coefficient of the second power of the unknown number.*

In particular, if the coefficient of the second power of the unknown number be 1, *the product of the roots is equal to the term free from the unknown number.*

E.g., the roots of the equation $6x^2 - x - 2 = 0$ are $\frac{2}{3}$ and $-\frac{1}{2}$; their sum is $\frac{1}{6}$ (the coefficient of x, with sign reversed, divided by the coefficient of x^2), and their product is $-\frac{1}{3}$ (the term free from x divided by the coefficient of x^2).

The roots of the equation $x^2 - 5x + 6 = 0$ are 2 and 3; their sum is 5, and their product is 6.

12. Formation of an Equation from its Roots.—The relations of the last article enable us to form an equation if its roots be given. We may always assume that the coefficient of the second power of the unknown number is 1.

Ex. 1. Form the equation whose roots are $-1, 2$.

We have $r_1 + r_2 = -1 + 2 = 1$, the coefficient of x, with sign reversed; and $r_1r_2 = -1 \times 2 = -2$, the term free from x.

Therefore the required equation is $x^2 - x - 2 = 0$.

Ex. 2. Form the equation whose roots are $1 + 2\sqrt{3}, 1 - 2\sqrt{3}$.

We have $r_1 + r_2 = (1 + 2\sqrt{3}) + (1 - 2\sqrt{3}) = 2$;

and $r_1r_2 = (1 + 2\sqrt{3})(1 - 2\sqrt{3}) = 1 - 12 = -11$.

Therefore the required equation is $x^2 - 2x - 11 = 0$.

Ex. 3. Form the equation whose roots are $2 + 5\sqrt{-3}$, $2 - 5\sqrt{-3}$.

We have $r_1 + r_2 = (2 + 5\sqrt{-3}) + (2 - 5\sqrt{-3}) = 4$;

and $r_1r_2 = (2 + 5\sqrt{-3})(2 - 5\sqrt{-3}) = 4 + 75 = 79$.

Therefore the required equation is $x^2 - 4x + 79 = 0$.

13. The roots of a quadratic equation, all of whose terms are in the first member, are the roots of the two linear factors into which this member can be resolved. Consequently a quadratic equation whose roots are given can be formed by multiplying together the two linear factors which (equated to 0) have as roots the given roots.

Ex. Form the equation whose roots are -1, 2.

Since -1 is the root of $x + 1 = 0$,

and 2 is the root of $x - 2 = 0$,

the required quadratic is $(x + 1)(x - 2) = 0$, or $x^2 - x - 2 = 0$.

When the roots are irrational or imaginary, the method of the preceding article is to be preferred.

14. One Root Known. — When one root of a given quadratic equation is known, the other root can be found without solving the equation.

Ex. 1. One root of the equation $x^2 - 5x + 6 = 0$ is 3; what is the other root?

Since 5 is the sum of the roots, the required root is $5 - 3, = 2$.

Or, since $x^2 - 5x + 6$ is the product of two linear factors, one of which is $x - 3$, the other factor is

$$\frac{x^2 - 5x + 6}{x - 3}, = x - 2.$$

The required root is therefore the root of $x - 2 = 0$, or $x = 2$.

Ex. 2. One root of the equation $x^2 + 2x - 1 = 0$, is $-1 + \sqrt{2}$; what is the other root?

The required root is $-2 - (-1 + \sqrt{2}), = -1 - \sqrt{2}$.

15. Expressions Symmetrical in the Roots. — The value of an expression which is symmetrical in the roots of a given quadratic equation can be found without solving the equation.

Ex. If r_1 and r_2 be the roots of $x^2 + px + q = 0$, find the value of $r_1^2 + r_2^2$.

We have $r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1r_2 = (-p)^2 - 2q = p^2 - 2q$.

16. We can sometimes form an equation whose roots have definite relations to the roots of a given quadratic equation without solving the latter.

Ex. Form an equation whose roots are the reciprocals of the roots of the equation $x^2 - 8x + 15 = 0$.

Let r_1 and r_2 be the roots of the given equation. Then $\frac{1}{r_1}$ and $\frac{1}{r_2}$ are the roots of the required equation.

We have $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{8}{15}$, the coefficient of x (with sign reversed) in the required equation; and $\frac{1}{r_1} \times \frac{1}{r_2} = \frac{1}{r_1 r_2} = \frac{1}{15}$, the term free from x in the required equation.

Consequently the required equation is

$$x^2 - \tfrac{8}{15}x + \tfrac{1}{15} = 0, \text{ or } 15x^2 - 8x + 1 = 0.$$

Nature of the Roots of a Quadratic Equation.

17. In many applications it is important to know, without having to solve an equation, the nature of its roots, *i.e.*, whether they are both *real and unequal*, whether they are both *real and equal*, whether they are *imaginary*, etc.

In the general solution

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}, \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

of the equation $ax^2 + bx + c = 0$,

a, b, and c are limited to real, rational values.

(i.) *The two roots are real and unequal when* $b^2 - 4ac$ *is positive*, i.e., *when* $b^2 - 4ac > 0$.

E.g., in $x^2 + 4x - 12 = 0$,

$$a = 1,\ b = 4,\ c = -12;\ \text{and since } b^2 - 4ac, = 16 + 48,$$

is *positive*, the roots of this equation are real and unequal.

(ii.) *The two roots are real and equal when* $b^2 - 4ac$ *is equal to* 0; i.e., *when* $b^2 = 4ac$.

E.g., in $x^2 - 4x + 4 = 0$,

$$a = 1,\ b = -4,\ c = 4;\ \text{and since } b^2 = 4ac,$$

the roots of this equation are real and equal.

(iii.) *The two roots are conjugate complex numbers when* $b^2 - 4ac$ *is negative*; i.e., *when* $b^2 - 4ac < 0$.

E.g., in $x^2 - 2x + 3 = 0$,

$$a = 1,\ b = -2,\ c = 3;\ \text{and since } b^2 - 4ac, = 4 - 12, = -8,$$

is *negative*, the roots of this equation are complex numbers.

(iv.) *The two roots are real and rational when* $b^2 - 4ac$ *is positive and the square of a rational number.*

E.g., in $$x^2 - 5x + 4 = 0,$$
$$a = 1,\ b = -5,\ c = 4;\ \text{and since } b^2 - 4ac, = 9,$$
is the square of a rational number, the roots of this equation are real and rational.

(v.) *The two roots are real, but conjugate irrationals, when* $b^2 - 4ac$ *is positive and not the square of a rational number.*

E.g., in $$x^2 - 5x + 2 = 0,$$
$$a = 1,\ b = -5,\ c = 2;\ \text{and since } b^2 - 4ac, = 25 - 8, = 17,$$
is positive and not the square of a rational number, the roots of this equation are conjugate irrational numbers.

(vi.) *The two roots are equal and opposite when* $b = 0$.

We then have $r_1 = \dfrac{\sqrt{(-4ac)}}{2a}$, and $r_2 = -\dfrac{\sqrt{(-4ac)}}{2a}$.

Notice that in this case the roots are real or imaginary according as ac is negative or positive.

E.g., from $2x^2 + 5 = 0$ we have $x = \sqrt{-\frac{5}{2}} = \pm \frac{1}{2}\sqrt{-10}$.

(vii.) *One root is zero when* $c = 0$.

We then have $r_1 = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2}}{2a} = -\dfrac{b}{2a} + \dfrac{b}{2a} = 0;$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2}}{2a} = -\frac{b}{2a} - \frac{b}{2a} = -\frac{b}{a}.$$

E.g., from $2x^2 - 3x = 0$, we have $x(2x - 3) = 0$;

whence $$x = 0, \text{ and } x = \tfrac{3}{2}.$$

(viii.) *Both roots are zero when* $b = 0$ *and* $c = 0$.

We then have $$r_1 = 0 + 0,\ r_2 = 0 - 0.$$

E.g., from $3x^2 = 0$ we obtain $x = 0$ and $x = 0$.

18. As long as a is not equal to 0, however near its value may be to 0, the values of r_1 and r_2 given above constitute the solution of the equation. If, then, we assume that these values still give the roots when $a = 0$, we must determine the nature of the roots under this assumption.

(i.) *One root is infinite and the other* $-\dfrac{c}{b}$, *when* $a = 0,\ b \neq 0,\ c \neq 0$.

For $$r_1 = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} = \frac{[-b + \sqrt{(b^2 - 4ac)}][-b - \sqrt{(b^2 - 4ac)}]}{2a[-b - \sqrt{(b^2 - 4ac)}]}$$
$$= \frac{b^2 - (b^2 - 4ac)}{2a[-b - \sqrt{(b^2 - 4ac)}]} = \frac{2c}{-b - \sqrt{(b^2 - 4ac)}}.$$

If in $\dfrac{2c}{-b - \sqrt{(b^2 - 4ac)}}$ we let $a = 0$, we obtain $r_1 = \dfrac{2c}{-b - b} = -\dfrac{c}{b}$.

Also, $$r_2 = \frac{-b-\sqrt{b^2}}{0} = \frac{-2b}{0} = \infty.$$

In this case we are apparently dealing with the linear equation
$$bx + c = 0,$$
and not with a quadratic. But in applications of Algebra it is frequently necessary to consider the coefficient of x^2 as growing smaller and smaller without limit, *i.e.*, as approaching 0.

The meaning of the results given above are that as a grows smaller and smaller without limit, one root grows larger and larger without limit, and the other root becomes more and more nearly equal to $-\frac{c}{b}$.

E.g., the equation $0 \cdot x^2 + 2x + 3 = 0$ has one root ∞ and one root $-\frac{3}{2}$.

(ii.) *Both roots are infinite when* $a = 0$, $b = 0$, $c \neq 0$.

We have $r_1 = \dfrac{2c}{-b-\sqrt{(b^2-4ac)}}$, as above.

And $$r_2 = \frac{-b-\sqrt{(b^2-4ac)}}{2a} = \frac{b^2-(b^2-4ac)}{2a[-b+\sqrt{(b^2-4ac)}]}$$
$$= \frac{2c}{-b+\sqrt{(b^2-4ac)}}.$$

If we now let $a = 0$, $b = 0$, $c \neq 0$, we obtain
$$r_1 = \frac{2c}{0} = \infty, \text{ and } r_2 = \frac{2c}{0} = \infty.$$

Attention is called to the remarks at the end of (i.).

E.g., the equation $0 \cdot x^2 + 0 \cdot x + 2 = 0$ has two infinite roots.

19. If, in simplifying a quadratic equation, the terms of the second degree in the unknown number be canceled, an infinite root is lost.

E.g., solve the equation $(1+2x)(2-3x) = 5-6x^2$.

Performing indicated operations, $2 + x - 6x^2 = 5 - 6x^2$.

Transferring and uniting terms, $x = 3$.

In canceling $-6x^2$ an infinite root was lost.

It was for this reason that in the principle for adding or subtracting the same number or expression to or from both members of an equation the roots were limited to finite values.

Maxima and Minima.

20. Pr. 1. What is the least value of $x^2 + 6x + 11$ for real values of x? What is the value of x which gives this least value?

Let $x^2 + 6x + 11 = y$. Then we are to find the least value of y for all possible real values of x.

We then have $$x^2 + 6x + 9 = y - 2;$$
or $$x + 3 = \sqrt{(y - 2)}.$$

Now x will be real only for values of $y =$ or > 2. That is, 2 is the least value of y, $= x^2 + 6x + 11$, for real values of x. When $y = 2$, $x = -3$.

Pr. 2. Between what bounds do the values of the fraction $\frac{x^2 + 4x - 8}{x - 2}$ lie for real values of x?

Let $$\frac{x^2 + 4x - 8}{x - 2} = y. \qquad (1)$$

Whence $$x^2 + (4 - y)x + (2y - 8) = 0.$$

Then, by Art. 17 (i.), the values of x will be real when
$$(4 - y)^2 - 4(2y - 8) > 0;$$
that is, when $y^2 - 16y + 48 > 0$, or $(y - 4)(y - 12) > 0$.

This expression will be greater than 0, for all values of $y < 4$, and for all values of $y > 12$.

That is, the given fraction can have all values between $-\infty$ and 4, and between 12 and $+\infty$.

The values of x corresponding to the bounds of the fraction are found by solving the equations obtained by equating the given fraction to 4 and 12 respectively. We then have:

When $y = 4$, $x = 0$; when $y = 12$, $x = 4$.

Pr. 3. Under what condition can
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
be resolved into linear rational factors?

Solving the given expression as a quadratic in x we obtain
$$ax + hy + g = \pm\sqrt{[(h^2 - ab)y^2 + 2(hg - af)y + (g^2 - ac)]}. \qquad (1)$$

Equation (1) gives two linear rational factors when the expression under the radical is a perfect square; that is, when
$$(hg - af)^2 - (h^2 - ab)(g^2 - ac) = 0;$$
or $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

EXERCISES VI.

Form the equations whose roots are:

1. 8, 2. **2.** -5, -3. **3.** 10, 10. **4.** 7, -3.

5. 4, -10. **6.** $2\frac{1}{2}$, $1\frac{3}{5}$. **7.** $-\frac{2}{3}$, $-1\frac{1}{2}$. **8.** $-\frac{1}{4}$, 8.

9. 2, 0. **10.** a, b. **11.** $-a$, -1. **12.** a^2, $-4a^2$.

13. $\frac{a}{b}$, $\frac{b}{a}$. **14.** $\frac{a+b}{a-b}$, 1. **15.** $\frac{a}{2a - 2b}$, $\frac{b}{2b - 2a}$.

16. $\sqrt{2}, -\sqrt{2}$. **17.** $5\sqrt{7}, -5\sqrt{7}$. **18.** $\frac{1}{2}\sqrt{-3}, -\frac{1}{2}\sqrt{-3}$.

19. $1+\sqrt{7}, 1-\sqrt{7}$. **20.** $\frac{1}{2}-\frac{1}{3}\sqrt{11}, \frac{1}{2}+\frac{1}{3}\sqrt{11}$.

21. $3-\sqrt{-5}, 3+\sqrt{-5}$. **22.** $\frac{2}{3}-\frac{1}{2}\sqrt{-1}, \frac{2}{3}+\frac{1}{2}\sqrt{-1}$.

Find the second root of each of the following equations, without solving the equation:

23. $x^2 - 9x + 20 = 0$, when $r_1 = 4$.

24. $6x^2 - x - 1 = 0$, when $r_1 = -\frac{1}{3}$.

25. $x^2 - (a+b)^2 = 0$, when $r_1 = a + b$.

26. $x^2 - (a^2 - b^2)x = 0$, when $r_1 = 0$.

27. $b^2x^2 + 2abx + a^2 = 0$, when $r_1 = -\dfrac{a}{b}$.

28. $(a^2 - b^2)x^2 + 4abx - a^2 + b^2 = 0$, when $r_1 = \dfrac{a-b}{a+b}$.

29. $x^2 - 2x - 1 = 0$, when $r_1 = 1 - \sqrt{2}$.

30. $x^2 + x\sqrt{5} + 1 = 0$, when $r_1 = -\frac{1}{2}(\sqrt{5} + 1)$.

31. $x^2 - 6x + 13 = 0$, when $r_1 = 3 + 2\sqrt{-1}$.

32. $(a+b)^2x^2 - (a+b)cx - ac = 0$, when $r_1 = \dfrac{c - \sqrt{(c^2 + 4ac)}}{2(a+b)}$.

If r_1 and r_2 stand for the roots of the equation $x^2 + px + q = 0$, express each of the following symmetrical expressions in terms of p and q:

33. $r_1^2 + r_2^2$. **34.** $r_1^3 + r_2^3$. **35.** $r_1^4 + r_2^4$.

36. $\dfrac{1}{r_1} + \dfrac{1}{r_2}$. **37.** $\dfrac{r_1}{r_2} + \dfrac{r_2}{r_1}$. **38.** $\dfrac{r_1 + r_2}{r_1} + \dfrac{r_1 + r_2}{r_2}$.

39–44. Express each of the relations given in Exx. 33–38 in terms of the roots of the equation $x^2 + x - 6 = 0$.

Without solving the equations $x^2 + px + q = 0$ and $x^2 - 10x + 40 = 0$, form the equations whose roots are

45. The opposites of the roots of the given equations.

46. The reciprocals of the roots of the given equations.

47. Twice the roots of the given equations.

48. The roots of the given equations increased by 2.

49. The roots of the given equations diminished by 5.

50. The squares of the roots of the given equations.

Without solving the following equations, determine the nature of the roots of each one:

51. $x^2 + 17x + 70 = 0$. **52.** $x^2 - 6x = 27$.

53. $x^2 + 12x = -40$. **54.** $x^2 - 6x + 9 = 0$.

55. $x^2 + 5x - 14 = 0$. **56.** $x^2 + 20x = -100$.

57. $x^2 - x = 12.$ **58.** $x^2 - 8x + 25 = 0.$

59. $x^2 - 13x + 22 = 0.$ **60.** $x^2 - 8x = 16.$

61. $9x^2 - 12x + 4 = 0.$ **62.** $8x^2 - 2x - 25 = 0.$

63. $16x^2 + 8x + 49 = 0.$ **64.** $10x^2 - 21x - 10 = 0.$

65. $16x^2 + 40x + 25 = 0.$ **66.** $25x^2 + 4x - 77 = 0.$

67. $20x^2 + 19x = -3.$ **68.** $4x^2 + 52x = 87.$

For what values of m are the roots of each of the following equations equal? For what values of m are the roots real and unequal? And for what values of m are the roots complex numbers?

69. $mx^2 + 4x + 1 = 0.$ **70.** $2x^2 + mx + 1 = 0.$

71. $3x^2 + 6x + m = 0.$ **72.** $mx^2 + mx + 1 = 0.$

Find the greatest, or the least, value of each of the following expressions for real values of x, and the corresponding value of x:

73. $x^2 - 6x + 7.$ **74.** $10 + 3x - x^2.$

75. $2x^2 - 5x + 7.$ **76.** $3 + 8x - 5x^2.$

Find the bounds of each of the following fractions for real values of x, and the corresponding values of x:

77. $\frac{x^2 + 3x - 3}{x - 1}.$ **78.** $\frac{x^2 + x + 2}{x + 2}.$ **79.** $\frac{4x^2 + 24x + 27}{13 - 8x}.$

80. $\frac{8x + 3}{4x^2 + 8x + 3}.$ **81.** $\frac{24x + 7}{4x^2 + 8x + 2}.$ **82.** $\frac{x + 3}{x^2 + 6x - 3}.$

Which of the following expressions can be separated into rational linear factors?

83. $3x^2 + 13xy - 10y^2 - 11x + 13y - 4.$

84. $5x^2 - 14xy - 3y^2 - 8x + 8y + 3.$

85. $2x^2 - 4xy - 6y^2 - 3x + 17y - 4.$

86. $4x^2 - 7xy + 3y^2 + 9x - 5y + 3.$

87. Divide 100 into two parts so that their product shall be a maximum.

88. Divide 20 into two parts so that the sum of their squares shall be a minimum.

89. Two points, A and B, move along two perpendicular lines. A is 13 feet and B is 16 feet from the point of intersection, P. A moves at the rate of 4 feet a second toward P, and B at the rate of 3 feet a second away from P. After how many seconds will A and B be the least distance from each other?

Problems.

21. Pr. 1. The sum of two numbers is 15 and their product is 56; what are the numbers?

Let x stand for one of the numbers; then, by the first condition, $15-x$ stands for the other number. By the second condition

$$x(15-x)=56; \text{ whence } x=7, \text{ and } 8.$$

Therefore $x=7$, one of the numbers, and $15-x=8$, the other number. Observe that if we take $x=8$, then $15-x=7$. That is, the two required numbers are the two roots of the quadratic equation.

Pr. 2. Divide 100 into two parts whose product is 2600.

Let x stand for the less part, and $100-x$ for the greater.

By the second condition, $x(100-x)=2600$. The roots of this equation are $50+10\sqrt{-1}$ and $50-10\sqrt{-1}$.

An imaginary result always indicates inconsistent conditions in the problem. The inconsistency of these conditions may be shown as follows:

Let d stand for the difference between the two parts of 100. Then $50+\frac{1}{2}d$ stands for the greater part, and $50-\frac{1}{2}d$ for the less.

The product of the two parts is

$$(50+\tfrac{1}{2}d)(50-\tfrac{1}{2}d), =2500-(\tfrac{1}{2}d)^2=2500-\tfrac{1}{4}d^2.$$

Since d^2 is positive for all *real* values of d, the product $2500-\frac{1}{4}d^2$ must be less than 2500. Consequently 100 cannot be divided into two parts whose product is greater than 2500.

22. When the solution of a problem leads to a quadratic equation, it is necessary to determine whether either or both of the roots of the equation satisfy the conditions expressed and implied in the problem.

Positive results, in general, satisfy all the conditions of the problem.

A *negative result*, as a rule, satisfies the conditions of the problem, when they refer to abstract numbers. When the required numbers refer to quantities which can be understood in opposite senses, as opposite directions, etc., an intelligible meaning can usually be given to a negative result.

An *imaginary result* always implies inconsistent conditions.

23. The interpretation of a negative result is often facilitated by the following principle:

If a given quadratic equation have a negative root, then the equation obtained from the given one by changing the sign of x has a positive root of the same absolute value.

Let $-r$ be a root of $ax^2 + bx + c = 0$. (1)

Then, since $-r$ must satisfy the equation, we have

$$a(-r)^2 + b(-r) + c = 0,$$

or

$$ar^2 - br + c = 0. \qquad (2)$$

But equation (2) shows that r satisfies the equation

$$ax^2 - bx + c = 0,$$

which is obtained from (1) by changing the sign of x.

Pr. A man bought muslin for $3.00. If he had bought three yards more for the same money, each yard would have cost him 5 cents less. How many yards did he buy?

Let x stand for the number of yards the man bought. Then 1 yard cost $\frac{300}{x}$ cents. If he had bought $x + 3$ yards for the same money, each yard would have cost $\frac{300}{x+3}$ cents.

Therefore $\frac{300}{x} - \frac{300}{x+3} = 5$; whence $x = 12$ and -15.

The root 12 satisfies the equation and also the conditions of the problem; the root -15 has no meaning.

But if x be replaced by $-x$ in the equation, we obtain a new equation

$$\frac{300}{-x} - \frac{300}{-x+3} = 5, \text{ or } \frac{300}{x-3} - \frac{300}{x} = 5, \qquad (2)$$

whose roots are -12 and $+15$.

Equation (2) evidently corresponds to the problem: A man bought muslin for $3.00. If he had bought 3 yards less for the same money, each yard would have cost him 5 cents more.

Notice that the intelligible result, 12, of the first statement has become -12 and is meaningless in the second statement.

Attention is called to the remarks in Ch. XII., Art. 6.

EXERCISES VII.

1. If 1 be added to the square of a number, the sum will be 50. What is the number?

2. If 5 be subtracted from a number, and 1 be added to the square of the remainder, the sum will be 10. What is the number?

3. One of two numbers exceeds 50 by as much as the other is less than 50, and their product is 2400. What are the numbers?

4. The product of two consecutive integers exceeds the smaller by 17,424. What are the numbers?

5. If 27 be divided by a certain number, and the same number be divided by 3, the results will be equal. What is the number?

6. What number, added to its reciprocal, gives 2.9?

7. What number, subtracted from its reciprocal, gives n? Let $n = 6.09$.

8. If n be divided by a certain number, the result will be the same as if the number were subtracted from n. What is the number? Let $n=4$.

9. If the product of two numbers be 176, and their difference be 5, what are the numbers?

10. A certain number was to be added to $\frac{1}{2}$, but by mistake $\frac{1}{2}$ was divided by the number. Nevertheless the correct result was obtained. What was the number?

11. If 100 marbles be so divided among a certain number of boys that each boy shall receive four times as many marbles as there are boys, how many boys are there?

12. The area of a rectangle, one of whose sides is 7 inches longer than the other, is 494 square inches. How long is each side?

13. The difference between the squares of two consecutive numbers is equal to three times the square of the less number. What are the numbers?

14. A merchant received $48 for a number of yards of cloth. If the number of dollars a yard be equal to three-sixteenths of the number of yards, how many yards did he sell?

15. In a company of 14 persons, men and women, the men spent $24 and the women $24. If each man spent $1 more than each woman, how many men and how many women were in the company?

16. A pupil was to add a certain number to 4, then to subtract the same number from 9, and finally to multiply the results. But he added the number to 9, then subtracted 4 from the number, and multiplied these results. Nevertheless he obtained the correct product. What was the number?

17. A man paid $80 for wine. If he had received 4 gallons less for the same money, he would have paid $1 more a gallon. How many gallons did he buy?

18. A man left $31,500 to be divided equally among his children. But since 3 of the children died, each remaining child received $3375 more. How many children survived?

19. Two bodies move from the vertex of a right angle along its sides at the rate of 12 feet and 16 feet a second respectively. After how many seconds will they be 90 feet apart?

20. A tank can be filled by two pipes, by the one in two hours less time than by the other. If both pipes be open $1\frac{7}{8}$ hours, the tank will be filled. How long does it take each pipe to fill the tank?

21. From a thread, whose length is equal to the perimeter of a square, 36 inches are cut off, and the remainder is equal in length to the perimeter of another square whose area is four-ninths of that of the first. What is the length of the thread?

22. A number of coins can be arranged in a square, each side containing 51 coins. If the same number of coins be arranged in two squares, the side of one square will contain 21 more coins than the side of the other. How many coins does the side of each of the latter squares contain?

23. A farmer wished to receive $2.88 for a certain number of eggs. But he broke 6 eggs, and in order to receive the desired amount he increased the price of the remaining eggs by $2\frac{2}{5}$ cents a dozen. How many eggs had he originally?

24. Two bodies move toward each other from A and B respectively, and meet after 35 seconds. If it takes the one 24 seconds longer than the other to move from A to B, how long does it take each one to move that distance?

25. It takes a boat's crew 4 hours and 12 minutes to row 12 miles down a river with the current, and back again against the current. If the speed of the current be 3 miles an hour, at what rate can the crew row in still water?

26. A man paid $300 for a drove of sheep. By selling all but 10 of them at a profit of $2.50 each, he received the amount he paid for all the sheep. How many sheep did he buy?

27. Two men start at the same time to go from A to B, a distance of 36 miles. One goes 3 miles more an hour than the other, and arrives at B 1 hour earlier. At what rate does each man travel?

28. It took a number of men as many days to dig a ditch as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

29. The front wheel of a carriage makes 6 revolutions more than the hind wheel in running 36 yards; if the circumference of each wheel were 1 yard longer, the front wheel would make but 3 revolutions more than the hind wheel in running the same distance. What is the circumference of each wheel?

30. Two men formed a partnership with a joint capital of $500. The first left his money in the business 5 months, and the second his money 2 months. Each realized $450, including invested capital. How much did each invest?

31. Two trains run toward each other from A and B respectively, and meet at a point which is 15 miles further from A than it is from B. After the trains meet, it takes the first train $2\frac{2}{3}$ hours to run to B, and the second train $3\frac{3}{8}$ hours to run to A. How far is A from B?

32. The perimeter of a rectangular lawn having around it a path of uniform width is 420 feet. The area of the lawn and path together exceeds twice the difference of their areas by 1200 square yards, and the width of the path is one-sixth of the shorter side of the lawn. Find the dimensions of lawn and path.

33. Water enters a forty-gallon cask through one pipe and is discharged through another. In 4 minutes one gallon more is discharged through the second pipe than enters through the first. The first pipe can fill the cask in 3 minutes less time than it takes the second to discharge 66 gallons. How long does it take the first pipe to fill the cask?

34. In a rectangle, whose sides are a and b inches respectively, a second rectangle is constructed. The sides of the inner rectangle are equally distant from the sides of the outer, and the area of the inner rectangle is one-nth of the remaining part of the outer. What are the lengths of the sides of the inner rectangle? Let $a = 70$, $b = 52\frac{1}{2}$, $n = 1$.

35. It has been found by experiment that when an object is removed to a point 2, 3, 4, ⋯ times its original distance from the source of light, its illumination is 2^2, 3^2, 4^2, ⋯ times as feeble. A lamp and a candle are 4 feet apart. At what point on the line joining them will the illumination from the candle be equal to that from the lamp, if the light of the lamp be 9 times as intense as that of the candle?

CHAPTER XXII.

EQUATIONS OF HIGHER DEGREE THAN THE SECOND.

We shall consider in this chapter a few higher equations which can be solved by means of quadratic equations.

1. **A Binomial Equation** is an equation of the form $x^n = a$, wherein n is a positive integer.

Certain binomial equations can be factored into linear and quadratic factors or factors which can be brought to quadratic form by proper substitutions.

Ex. 1. Solve the equation $x^3 - 1 = 0.$ (1)

Factoring, $(x-1)(x^2+x+1) = 0.$ (2)

This equation is equivalent to the two equations

$$x - 1 = 0, \text{ whence } x = 1;$$

and $$x^2 + x + 1 = 0, \text{ whence } x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}.$$

This example gives the three cube roots of 1, since $x^3 - 1 = 0$ is equivalent to $$x^3 = 1, \text{ or } x = \sqrt[3]{1}.$$

Therefore the three cube roots of 1 are $1, -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}$.

In general, the three cube roots of any number can be found by multiplying the principal cube root of the number in turn by the three algebraic cube roots of 1.

E.g., $$\sqrt[3]{8} = 2\sqrt[3]{1} = 2, \ -1 \pm \sqrt{-3};$$
the three cube roots of a are $\sqrt[3]{a}$, $\sqrt[3]{a}(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3})$, wherein $\sqrt[3]{a}$ denotes the principal cube root of a.

Ex. 2. Solve the equation $x^4 + 1 = 0$.

Factoring, $(x^2 + 1 + x\sqrt{2})(x^2 + 1 - x\sqrt{2}) = 0.$

This equation is equivalent to the two equations

$$x^2 + 1 + x\sqrt{2} = 0, \text{ whence } x = \tfrac{1}{2}\sqrt{2}(-1 \pm \sqrt{-1});$$

and $$x^2 + 1 - x\sqrt{2} = 0, \text{ whence } x = \tfrac{1}{2}\sqrt{2}(1 \pm \sqrt{-1}).$$

Since the given equation is equivalent to $x^4 = -1$, or $x = \sqrt[4]{-1}$, we conclude that the four fourth roots of -1 are

$$\tfrac{1}{2}\sqrt{2}(-1 \pm \sqrt{-1}), \ \tfrac{1}{2}\sqrt{2}(1 \pm \sqrt{-1})$$

The four fourth roots of any negative number can be found by multiplying the principal fourth root of the radicand *taken positively* in turn by the four fourth roots of -1.

E.g., $\sqrt[4]{-16} = 2\sqrt[4]{-1} = \sqrt{2}(-1 \pm \sqrt{-1}),\ \sqrt{2}(1 \pm \sqrt{-1})$.

2. Ex. 1. Solve the equation $x^4 - 9 = 2x^2 - 1$.

Since $x^4 = (x^2)^2$, we may take x^2 as the unknown number and solve this equation as a quadratic in x^2.

We then have $(x^2)^2 - 2x^2 - 8, = 0.$

Factoring, $(x^2 - 4)(x^2 + 2) = 0.$

Whence $x^2 - 4 = 0$, or $x = \pm 2$; and $x^2 + 2 = 0$, or $x = \pm\sqrt{-2}$.

In general, any equation containing only two powers of the unknown number, *one of which is the square of the other*, can be solved as a quadratic equation.

Ex. 2. Solve the equation $x^6 - 3x^3 = 40$.

Since $x^6 = (x^3)^2$, we take x^3 as the unknown number.

We then have $(x^3)^2 - 3x^3 = 40.$

Solving this equation for x^3, we obtain

$$x^3 = 8, \text{ whence } x = \sqrt[3]{8};\ \text{ and } x^3 = -5, \text{ whence } x = -\sqrt[3]{5}.$$

Therefore, by Art. 1, Ex. 1, the six roots of the given equation are

$$2,\ -1 \pm \sqrt{-3},\ -\sqrt[3]{5},\ \tfrac{1}{2}\sqrt[3]{5}(1 \mp \sqrt{-3}),$$

wherein $\sqrt[3]{5}$ denotes the principal cube root of 5.

Ex. 3. Solve the equation $(x^2 - 3x + 1)^2 = 6 + 5(x^2 - 3x + 1)$.

In this example $x^2 - 3x + 1$ is regarded as the unknown number, and may temporarily be represented by the letter y. The equation then becomes

$$y^2 = 6 + 5y;\ \text{ whence } y = 6, \text{ and } -1.$$

We therefore have the two equations

$$x^2 - 3x + 1 = 6, \text{ whence } x = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{29};$$

$$x^2 - 3x + 1 = -1, \text{ whence } x = 2,\ x = 1.$$

Therefore the roots of the given equation are $\frac{3}{2} \pm \frac{1}{2}\sqrt{29}$, 2, 1.

Attention is called to the fact that, in each example, we have obtained as many roots as there are units in the degree of the equation.

Frequently equations which do not at first appear to come under this case can, by a proper arrangement of terms, be made to do so.

Ex. 4. Solve the equation $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

The given equation can be written

$$x^4 + 2x^3 + x^2 - 8x^2 - 8x + 12 = 0,$$

or $$(x^2 + x)^2 - 8(x^2 + x) + 12 = 0.$$

If we now let $x^2 + x = y$, we have

$$y^2 - 8y + 12 = 0;\ \text{whence } y = 2, \text{ and } 6.$$

We then have to solve the equations

$$x^2 + x = 2\ (1),\ \text{and } x^2 + x = 6\ (2).$$

The roots of (1) are 1, -2; and the roots of (2) are 2, -3.

3. Ex. Solve the equation $\frac{x^2 + x + 1}{x^2 - x + 2} + \frac{x^2 - x + 2}{x^2 + x + 1} = 2\frac{1}{6}$.

If we let $\frac{x^2 + x + 1}{x^2 - x + 2} = y$, the given equation becomes $y + \frac{1}{y} = 2\frac{1}{6}$.

The roots of this equation are $\frac{3}{2}$, $\frac{2}{3}$.

We now have to solve the two equations

$$\frac{x^2 + x + 1}{x^2 - x + 2} = \frac{3}{2}\ (1),\ \text{and } \frac{x^2 + x + 1}{x^2 - x + 2} = \frac{2}{3}\ (2).$$

The roots of (1) are found to be 4, 1; and the roots of (2) are found to be $-\frac{5}{2} \pm \frac{1}{2}\sqrt{29}$.

An equation can be solved by this method when it contains only two expressions in the unknown number, one of which is the reciprocal of the other, and when the numerators and denominators of these expressions are of degree not higher than the second.

EXERCISES.

Solve each of the following equations:

1. $x^2 + 1 = 0$. **2.** $(x - 1)^3 = 8$. **3.** $(x + 2)^3 + 4 = 0$.

4. $(x+1)^3 = (3-x)^3$. **5.** $x^3 = (2a - x)^3$. **6.** $x^4 - 1 = 0$.

7. $(x + 1)^4 = 16$. **8.** $x^4 + 625(x+1)^4 = 0$. **9.** $x^6 + 1 = 0$.

10. $x^6 - 1 = 0$. **11.** $x^4 + 9 = 10x^2$. **12.** $x^4 - 6x^2 = -1$.

13. $(x^2 - 9)(x^2 - 16) = 15x^2$. **14.** $(x^2 - 10)(x^2 - 18) = 13x^2$.

15. $x^6 - 65x^3 = -64$. **16.** $x^8 + 5x^4 = 6$.

17. $\frac{(a + x)^4 + (a - x)^4}{(a + x)^3 + (a - x)^3} = 2a$. **18.** $\frac{x^4 + 6x^2 + 1}{x^4 - 6x^2 + 1} = \frac{3}{2}$.

19. $(x - 2)^6 - 19(x - 2)^3 = 216$. **20.** $(3x^2 - 5x + 1)^2 - 9x^2 + 15x = 7$.

21. $15x^2 - 35x - 3(7x - 3x^2 + 8)^2 + 310 = 0$.

22. $(x^2 - x + 1)^2 = 3x(x - 1) + 1$. **23.** $\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}$.

24. $\frac{x^2 - a^2}{x^2 + a^2} + \frac{x^2 + a^2}{x^2 - a^2} = \frac{34}{15}$. **25.** $\frac{x^2 - 5x + 3}{x^2 + 5x - 3} - \frac{x^2 + 5x - 3}{x^2 - 5x + 3} = \frac{8}{3}$.

26. $x^3 + x^2 - x - 1 = 0$. **27.** $3x^3 - 13x^2 + 13x - 3 = 0$.

28. $x^4 - 2x^3 + 2x^2 - 2x + 1 = 0$. **29.** $2x^4 - 5x^3 + 4x^2 - 5x + 2 = 0$.

CHAPTER XXIII.

IRRATIONAL EQUATIONS.

1. An **Irrational Equation** is an equation whose members are irrational in the unknown number or numbers; as, $\sqrt{(x+1)} = 3$.

Notice that we cannot speak of the *degree* of an irrational equation.

2. The solution of an irrational equation depends upon the principle: *If both members of an equation be raised to the same positive integral power, the resulting equation will have as roots the roots of the given equation, and, in general, additional roots.*

Let $$M = N$$ be the given equation.

Squaring both members, $$M^2 = N^2.$$

Whence $$M^2 - N^2 = 0, \text{ or } (M - N)(M + N) = 0.$$

This equation is equivalent to the two equations

$$M - N = 0, \text{ or } M = N, \text{ the given equation;}$$

and $$M + N = 0, \text{ or } M = -N, \text{ an additional equation.}$$

That is, the equation obtained by squaring both members of the given equation is equivalent to the given equation and an additional equation which differs from the given one in the sign of one of its members.

In like manner the principle can be proved for any positive integral powers of the members of the given equation.

E.g., if both members of the equation

$$x + 1 = 2$$

be squared, we have $(x+1)^2 = 4$; whence $x = 1$, and -3.

The root 1 satisfies the given equation; the root -3 is a root of the equation

$$x + 1 = -2,$$

which was introduced by squaring, and does not satisfy the given equation.

3. To solve an irrational equation, we must first derive from it a rational, integral equation. This step, which is usually effected by raising both members of the equation to the same positive integral power one or more times, is called *rationalizing the equation.*

In the following examples the roots will be limited to *principal* values:

Ex. 1. Solve the equation $x+\sqrt{(25-x^2)}=7$.

Before squaring, it is better to have the radical by itself in one member.

Transferring x, $$\sqrt{(25-x^2)}=7-x. \quad (1)$$

Squaring, $$25-x^2=49-14x+x^2. \quad (2)$$

The roots of this equation are 3, 4.

Both roots of (2) satisfy the given equation, since $3+\sqrt{(25-9)}=7$, and $4+\sqrt{(25-16)}=7$. Therefore no root was introduced by squaring both members of the given equation. This is also evident from the following considerations:

Any root of the additional equation,

$$\sqrt{(25-x^2)}=-(7-x), \text{ or } -\sqrt{(25-x^2)}=7-x, \quad (3)$$

obtained by changing the sign of one of the members of the given equation when prepared for squaring, must be a root of the rational integral equation (2). But both roots of this equation, 3 and 4, make the first member of (3) negative, and the second member positive. That is, equation (3) is an impossible equation.

Ex. 2. Solve the equation $x-\sqrt{(25-x^2)}=1$.

Transferring x, $$-\sqrt{(25-x^2)}=1-x. \quad (1)$$

Squaring $$25-x^2=1-2x+x^2. \quad (2)$$

The roots of this equation are 4 and -3.

The number 4 is a root of the given equation, since

$$4-\sqrt{(25-16)}=1;$$

but the number -3 is not a root of the given equation, since

$$-3-\sqrt{(25-9)}=-7, \text{ not } 1.$$

Therefore, the root -3 is a root of the additional equation

$$-\sqrt{(25-x^2)}=-(1-x), \text{ or } \sqrt{(25-x^2)}=1-x,$$

introduced by squaring.

That -3 is not a root of the given equation is also evident from the form of the equation. For any real value of x which makes $x-\sqrt{(25-x^2)}$ equal to 1 must be greater than 1, and therefore cannot be equal to -3.

The preceding examples illustrate the following method of solving irrational equations:

Transform the given equation so that one radical stands by itself in one member of the equation.

Equate equal powers of the two members when so transformed.

Repeat this process until a rational equation is obtained.

4. In the preceding article the indicated roots in the equations were limited to principal values.

At the same time an irrational equation, if written arbitrarily, may be inconsistent with the laws governing the relations between numbers. In such a case the equation is *impossible*, that is, it cannot be satisfied by either real or imaginary values of the unknown numbers.

E.g., $\sqrt{(x+6)}+\sqrt{(x+1)}=1$ is an impossible equation.

For it cannot be satisfied by any complex value of x, since by Ch. XX., Arts. 31 and 22, $\sqrt{(x+6)}+\sqrt{(x+1)}$ must be complex if x be complex, and hence cannot be equal to 1.

It cannot be satisfied by any *real positive* value of x, since, in that case, either $\sqrt{(x+1)}$ or $\sqrt{(x+6)}$ is greater than 1.

It cannot be satisfied by any *real negative* value of x, since, if x be negative and its absolute value be less than 1, $\sqrt{(x+6)}$ will be greater than 1, and if x be negative and its absolute value be greater than 1, $\sqrt{(x+1)}$ will be imaginary.

5. But if the restriction to *principal* roots be removed, any irrational equation contains in itself the statements of two or more equations.

E.g., if both *positive* and *negative* square roots be admitted, the equation

$$\sqrt{(x+6)}+\sqrt{(x+1)}=1$$

is equivalent to the four equations

$$\sqrt{(x+6)}+\sqrt{(x+1)}=1 \quad (1), \qquad \sqrt{(x+6)}-\sqrt{(x+1)}=1 \quad (2),$$

$$-\sqrt{(x+6)}+\sqrt{(x+1)}=1 \quad (3), \qquad -\sqrt{(x+6)}-\sqrt{(x+1)}=1 \quad (4),$$

in which the roots are limited to *principal* values.

The same rational integral equation will evidently be derived by rationalizing any one of these equations. Therefore the roots of this rational equation must comprise the roots of these four irrational equations. Consequently, in solving an irrational equation, we must expect to obtain not only its roots, but also the roots of the other three equations obtained by changing the signs of the radicals in all possible ways. Some of these equations can be rejected at once as impossible. The roots of the other irrational equations will be the roots of the rational equation. Thus, of the above equations, (1), (3), and (4) can be rejected at once as impossible.

The rational equation derived from any one of the four equations is

$$x+1=4\text{; whence } x=3.$$

The number 3 is a root of the one equation not rejected, since

$$\sqrt{(3+6)}-\sqrt{(3+1)}=1.$$

The same conclusions could have been reached by substituting the roots of the integral equation successively in the irrational equations, rejecting those which are not satisfied by any root.

Special Devices.

6. Ex. 1. Solve the equation

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16.$$

Since $$-3x^2+2x=-(3x^2-2x+4)+4,$$

we may take $\sqrt{(3x^2-2x+4)}$ as the unknown number, replacing it temporarily by y. We then have

$$y-y^2+4=-16.$$

The roots of this equation are 5, and -4.

Equating $\sqrt{(3x^2-2x+4)}$ to each of these roots, we have

$$\sqrt{(3x^2-2x+4)}=5, \text{ whence } x=3, -\tfrac{7}{3}.$$

$$\sqrt{(3x^2-2x+4)}=-4, \text{ whence } x=\tfrac{1}{3}(1\pm\sqrt{37}).$$

The numbers 3, $-\frac{7}{3}$ satisfy the given equation, and are therefore roots of that equation. The numbers $\frac{1}{3}\sqrt{(1\pm\sqrt{37})}$ do not satisfy the given equation.

But if the value of the radical be not restricted to the principal root, the given equation comprises the two equations

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16, \quad (1)$$

$$-\sqrt{(3x^2-2x+4)}-3x^2+2x=-16. \quad (2)$$

Then $\frac{1}{3}(1\pm\sqrt{37})$ are roots of (2).

Ex. 2. Solve the equation $\sqrt[4]{(3x^2+13)}+\sqrt{(3x^2+13)}=6$.

Assuming $\sqrt[4]{(3x^2+13)}$ as the unknown number, and representing it by y, we have $$y+y^2=6.$$

The roots of this equation are 2 and -3.

Equating $\sqrt[4]{(3x^2+13)}$ to each of these roots, we have

$$\sqrt[4]{(3x^2+13)}=2, \text{ whence } x=\pm 1,$$

$$\sqrt[4]{(3x^2+13)}=-3, \text{ whence } x=\pm\sqrt{\tfrac{68}{3}}=\pm\tfrac{2}{3}\sqrt{51}.$$

The numbers ± 1 are roots of the given equation, since $\sqrt[4]{16}+\sqrt{16}=6$.

The numbers $\pm\frac{2}{3}\sqrt{51}$ are evidently not roots of the given equation, but are found to be roots of the equation

$$-\sqrt[4]{(3x^2+13)}+\sqrt{(3x^2+13)}=6.$$

The preceding examples illustrate the following principle:

If a radical equation contain one radical, and an expression which is equal to the radicand or which can be made to differ from the radicand (or a multiple of the radicand) by a constant term, it can be solved as a quadratic equation. The same is true if the equation contain two radicals, one the square of the other, and in addition only constant terms. In both cases, the radicand must, in general, be a linear or a quadratic expression.

7. Irrational equations containing cube and higher roots in general lead to rational, integral equations of a higher degree than the second, and therefore cannot be solved by means of quadratic equations. But in some cases their solutions can be effected by special devices.

Ex. Solve the equation $\sqrt[3]{(8x+4)} - \sqrt[3]{(8x-4)} = 2$.

Cubing,

$$8x + 4 - 3[\sqrt[3]{(8x+4)}]^2\sqrt[3]{(8x-4)} + 3\sqrt[3]{(8x+4)}[\sqrt[3]{(8x-4)}]^2 - 8x + 4 = 8. \qquad (1)$$

Transferring and uniting terms, and dividing by -3,

$$[\sqrt[3]{(8x+4)}]^2\sqrt[3]{(8x-4)} - \sqrt[3]{(8x+4)}[\sqrt[3]{(8x-4)}]^2 = 0. \qquad (2)$$

Factoring, $\sqrt[3]{(8x+4)}\sqrt[3]{(8x-4)}[\sqrt[3]{(8x+4)} - \sqrt[3]{(8x-4)}] = 0. \qquad (3)$

This equation is equivalent to the three equations

$$\sqrt[3]{(8x+4)} = 0, \text{ whence } x = -\tfrac{1}{2}; \qquad (4)$$

$$\sqrt[3]{(8x-4)} = 0, \text{ whence } x = \tfrac{1}{2}; \qquad (5)$$

and $\sqrt[3]{(8x+4)} - \sqrt[3]{(8x-4)} = 0$,

whence $8x + 4 = 8x - 4. \qquad (6)$

Equation (6) is not satisfied by any finite value of x.

The numbers $-\tfrac{1}{2}$ and $\tfrac{1}{2}$ are found to satisfy the given equation.

EXERCISES.

Solve each of the following equations, and check the results. If a result does not satisfy an equation as written, determine what signs the radical terms must have in order that the result may satisfy the equation.

1. $\sqrt{(3x+4)} - 4 = 0$.
2. $\sqrt{(16+x)} = 2\sqrt{(x+6)}$.
3. $\sqrt{[5+\sqrt{(x-4)}]} = 3$.
4. $\sqrt[3]{(10x+35)} - 1 = 4$.
5. $\sqrt{(x^2-9)} = 4$.
6. $4x = 3\sqrt{(2x^2-4)}$.
7. $3 - \sqrt{(3x^2-4x+9)} = 0$.
8. $2 - \sqrt{(3x^2-11x)} = 0$.
9. $5x = 2\sqrt{(3x^2-x+15)}$.
10. $\sqrt[6]{(1\frac{1}{3}x+8)} - \sqrt[3]{4} = 0$.
11. $\sqrt{(4x+9)} - 2\sqrt{x} = 1$.
12. $\sqrt{[(x-5)-7+\sqrt{(x-12)}]} = 0$.
13. $\dfrac{x-1}{\sqrt{x}+1} = 4 + \dfrac{\sqrt{x}-1}{2}$.
14. $\dfrac{x+\sqrt{(x^2+7)}}{28} = \dfrac{1}{\sqrt{(x^2+7)}}$.
15. $\dfrac{2x+\sqrt{(4x^2-1)}}{2x-\sqrt{(4x^2-1)}} = 4$.
16. $\sqrt{\left(\dfrac{4}{x^2}+5\right)} - \sqrt{\left(\dfrac{4}{x^2}-5\right)} = 2$.
17. $\sqrt{[4x^2-\sqrt[3]{(3x-5)}]} = 2x$.
18. $\sqrt{[4x-\sqrt{(2x+3)}]} = 3$.
19. $\dfrac{\sqrt{x}-2}{\sqrt{x}+3} = \dfrac{\sqrt{x}+1}{\sqrt{x}+21}$.
20. $\dfrac{\sqrt{(3x+1)}+\sqrt{(3x)}}{\sqrt{(3x+1)}-\sqrt{(3x)}} = 2$.
21. $\sqrt{(2+x)} + \sqrt{x} = \dfrac{4}{\sqrt{(2+x)}}$.
22. $\dfrac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}} = \dfrac{5}{11}$.

23. $3x - 2\sqrt{x} - 1 = 0.$

24. $\sqrt{(x+2)} - \sqrt{(x^2 + 2x)} = 0.$

25. $7\sqrt{x} = 3\sqrt{(x^2 + 3x - 59)}.$

26. $x + 5 - \sqrt{(x+5)} = 6.$

27. $(5 - \sqrt{x})^2 = 2(7 + \sqrt{x}).$

28. $x - 7\sqrt{(51 - x)} = 33.$

29. $4\sqrt{(75 - x)} = x - 54.$

30. $(\sqrt[4]{x} - 3)^2 + (\sqrt[4]{x} - 2)^2 = 1.$

31. $\sqrt{(x-2)} + 2\sqrt{(x+3)} - 2\sqrt{(3x-2)} = 0.$

32. $\sqrt{(2x+9)} + \sqrt{(3x-15)} = \sqrt{(7x+8)}.$

33. $x^2 + \sqrt{[4x^2 + \sqrt{(16x^2 + 12x)}]} = (x+1)^2.$

34. $\sqrt{\dfrac{3x-4}{x-5}} + \sqrt{\dfrac{x-5}{3x-4}} = \dfrac{5}{2}.$

35. $\sqrt{\dfrac{3x+6}{7x-3}} + \sqrt{\dfrac{7x-3}{3x+6}} = \dfrac{13}{6}.$

36. $\dfrac{1}{\sqrt{(x+2)}} + \dfrac{1}{\sqrt{(3x-2)}} = \dfrac{4}{\sqrt{(8x^2 + 4x - 4)}}.$

37. $\dfrac{1}{x - \sqrt{(2 - x^2)}} + \dfrac{1}{x + \sqrt{(2 - x^2)}} = 1.$

38. $3x - x\sqrt{x} = 2\sqrt{x}.$

39. $\sqrt{x} + \sqrt[4]{x^3} = 2\sqrt[4]{x}.$

40. $x^2 - x + 2\sqrt{(x^2 - x - 11)} = 14.$

41. $x^2 + 24 = 2x + 6\sqrt{(2x^2 - 4x + 16)}.$

42. $\sqrt{(2x^2 - 3x + 5)} + 2x^2 - 3x = 1.$

43. $2x\sqrt{(4x^2 - 27x)} = -5x^2 + 27x + 9.$

44. $\sqrt{(3x^2 + 7x - 1)} - \sqrt{(3x^2 - 4x + 5)} = 8.$

45. $\sqrt{(2x^2 - 7x + 7)} + \sqrt{(2x^2 + 9x - 1)} = 6.$

46. $\sqrt[3]{x} + \sqrt[3]{(28 - x)} = 4.$

47. $\sqrt[3]{(1 + \sqrt{x})} = 2 - \sqrt[3]{(1 - \sqrt{x})}.$

48. $\sqrt[3]{(14 + x)} + \sqrt[3]{(14 - x)} = 4.$

49. $\sqrt[4]{(41 + x)} + \sqrt[4]{(41 - x)} = 4.$

50. $\sqrt[4]{(a+x)} + \sqrt[4]{(a-x)} = \sqrt[4]{(2a)}.$

51. $\sqrt{[\sqrt{(cx + a^2)} - a]} = c.$

52. $\sqrt{\dfrac{a^2 + x^2}{a^2 - x^2}} = \dfrac{a}{b}.$

53. $\dfrac{\sqrt{x} + \sqrt{b}}{\sqrt{x} - \sqrt{b}} = \dfrac{a}{b}.$

54. $\sqrt{\left(\dfrac{a^2}{x} + b\right)} - \sqrt{\left(\dfrac{a^2}{x} - b\right)} = c.$

55. $\dfrac{1 + \sqrt{(1-x)}}{1 - \sqrt{(1-x)}} = n.$

56. $\sqrt{(a+x)} + \sqrt{(a-x)} = \dfrac{a}{\sqrt{(a+x)}}.$

57. $\dfrac{1}{x} + \dfrac{1}{a} = \sqrt{\left[\dfrac{1}{a^2} - \sqrt{\left(\dfrac{1}{a^2x^2} + \dfrac{1}{x^4}\right)}\right]}.$

58. $\dfrac{x^2}{a - \sqrt{(a^2 - x^2)}} - \dfrac{x^2}{a + \sqrt{(a^2 - x^2)}} = a.$

59. $\sqrt{(1 - x + x^2)} + \sqrt{(1 + x + x^2)} = m.$

60. $\dfrac{a\sqrt{(x-b)} + b\sqrt{(a-x)}}{\sqrt{(a-x)} + \sqrt{(x-b)}} = x.$

61. $\dfrac{a + x + \sqrt{(a^2 - x^2)}}{a + x - \sqrt{(a^2 - x^2)}} = \dfrac{b}{x}.$

62. $\dfrac{a+x}{\sqrt{x} + \sqrt{(a+x)}} - \dfrac{a+x}{\sqrt{a} - \sqrt{(a+x)}} = 0.$

63. $\dfrac{3\sqrt[3]{(x-5)} + 5\sqrt[3]{(3-x)}}{\sqrt[3]{(3-x)} + \sqrt[3]{(x-5)}} = x.$

64. $\dfrac{\sqrt{a} - \sqrt{[a - \sqrt{(a^2 - ax)}]}}{\sqrt{a} + \sqrt{[a - \sqrt{(a^2 - ax)}]}} = b.$

CHAPTER XXIV.

SIMULTANEOUS QUADRATIC AND HIGHER EQUATIONS.

To obtain a definite solution of a system of two or more quadratic or higher equations, as many equations must be given as there are unknown numbers. As in linear systems, the given equations must be consistent and independent.

The solution of a system of quadratic or higher equations in general involves the solution of an equation of higher degree than the second, and therefore cannot be effected by the methods for solving quadratic equations. But there are many special systems whose solutions can be made to depend upon the solutions of quadratic equations.

§ 1. SIMULTANEOUS QUADRATIC EQUATIONS.

1. Elimination by Substitution. — When one equation of a system of two equations is of the first degree, the solution can be obtained by the method of substitution.

Ex. Solve the system
$$\left.\begin{aligned} y + 2x &= 5, \\ x^2 - y^2 &= -8. \end{aligned}\right\} \qquad \begin{aligned} (1) \\ (2) \end{aligned}$$

Solving (1) for y, $$y = 5 - 2x. \qquad (3)$$

Substituting $5 - 2x$ for y in (2),
$$x^2 - 25 + 20x - 4x^2 = -8. \qquad (4)$$

From this equation we obtain $$x = 1, \qquad (5)$$
$$x = 5\tfrac{2}{3}. \qquad (6)$$

Substituting 1 for x in (3), $y = 3$.

Substituting $5\frac{2}{3}$ for x in (3), $y = -6\frac{1}{3}$.

The system (1), (2) is equivalent to the system (3), (4), which is equivalent to the two systems (3), (5) and (3), (6).

Therefore the solutions of the given system are 1, 3; $5\frac{2}{3}$, $-6\frac{1}{3}$, the first number of each pair being the value of x, and the second the corresponding value of y.

Had we substituted 1 for x in (2), we should have obtained $y = \pm 3$.

But the solution 1, -3 does not satisfy equation (1).

By the principle of equivalent equations, proved in Ch. XIII., § 2, Art. 2 (iii.), equation (3), obtained from (1) by solving for y, and equation (4), obtained by substituting this value for y in (2), form a system equivalent to the given system. This principle does not, however, prove that (2) and (4) are necessarily equivalent to the given system. In this example, since (2) and (4) give more solutions than (1) and (4), the system formed by (2) and (4) cannot be equivalent to the given system. Therefore, having obtained the values of one of the unknown numbers, we should obtain the values of the other by substituting in the equation of the first degree. This advice was unnecessary in solving systems of linear simultaneous equations, since then both equations were of the first degree.

EXERCISES I.

Solve each of the following systems:

1. $\begin{cases} xy = 54, \\ 3x = 2y. \end{cases}$ **2.** $\begin{cases} 4x - 3y = 24, \\ xy = 96. \end{cases}$ **3.** $\begin{cases} 2x^2 - 3y^2 = 24, \\ 2x = 3y. \end{cases}$

4. $\begin{cases} 3x - 2y = 1, \\ x^2 + y^2 = 74. \end{cases}$ **5.** $\begin{cases} 2x + 3y = 10, \\ x(x + y) = 25. \end{cases}$ **6.** $\begin{cases} 4x^2 - xy = 0, \\ 2x - 3y = 6. \end{cases}$

7. $\begin{cases} x^2 + xy + y^2 = 343, \\ 2x - y = 21. \end{cases}$ **8.** $\begin{cases} 2x^2 - 3xy + y^2 = 14, \\ 2x - y = 7. \end{cases}$

9. $\begin{cases} (x - 7)(y + 3) = 48, \\ x + y = 18. \end{cases}$ **10.** $\begin{cases} (x - 3)(y - 4) = -6, \\ 4x + 3y = 10. \end{cases}$

11. $\begin{cases} 2x - 3y = 11, \\ \dfrac{4}{x} - \dfrac{3}{y} = -\dfrac{17}{7}. \end{cases}$ **12.** $\begin{cases} x + 2y = 1, \\ \dfrac{x}{y} + \dfrac{y}{x} + 3\frac{1}{3} = 0. \end{cases}$

13. $\begin{cases} \dfrac{x-1}{y-1}=2, \\ \dfrac{x^2+x+1}{y^2+y+1}=\dfrac{19}{7}. \end{cases}$

14. $\begin{cases} \dfrac{x+1}{y+1}=\dfrac{3}{2}, \\ \dfrac{x^2+y}{x+y^2}=\dfrac{32}{15}. \end{cases}$

15. $\begin{cases} 5\,xy+2\,y^2+4\,x+16=0, \\ 11\,x+5\,y=4. \end{cases}$

16. $\begin{cases} x^2+xy+5\,x+10\,y=29, \\ x+2\,y=3. \end{cases}$

2. Elimination by Addition and Subtraction. — When both equations of a system of two quadratic equations contain only the squares of the unknown numbers, the solution can be obtained by the method of addition and subtraction.

Ex. 1. Solve the system $9\,x^2-8\,y^2=28,$ (1)

$7\,x^2+3\,y^2=31.$ (2)

We will first eliminate y^2.

Multiplying (1) by 3, $27\,x^2-24\,y^2=84.$ (3)

Multiplying (2) by 8, $56\,x^2+24\,y^2=248.$ (4)

Adding (3) and (4), $83\,x^2=332.$ (5)

Whence $x=2,$ (6)

and $x=-2.$ (7)

Substituting 2 for x in (1), $y=\pm 1.$ (8)

Substituting -2 for x in (1), $y=\pm 1.$ (9)

The given system is equivalent to the system (3), (4), which is equivalent to the system (5), (1); this last system is equivalent to the two systems (6), (1) and (7), (1).

The solutions of the system (6), (1) are 2, 1; 2, -1.

The solutions of the system (7), (1) are -2, 1; -2, -1.

Therefore, the given system has the four solutions:

$$2, 1;\ 2, -1;\ -2, 1;\ -2, -1.$$

Many other examples are most easily solved by this method.

Ex. 2. Solve the system $x^2+3\,y=18,$ (1)

$2\,x^2-5\,y=3.$ (2)

We will first eliminate y.

Multiplying (1) by 5, $5\,x^2+15\,y=90.$ (3)

Multiplying (2) by 3, $6\,x^2-15\,y=9.$ (4)

Adding (3) and (4), $11\,x^2=99.$

Whence, $x = 3$, and $x = -3$.
Substituting 3 for x in (1), $y = 3$.
Substituting -3 for x in (1), $y = 3$.

The given system has the two solutions 3, 3; -3, 3.

Notice that this example could also have been solved by the method of substitution.

EXERCISES II.

Solve each of the following systems:

1. $\begin{cases} x^2 + y^2 = 13, \\ x^2 - y^2 = 5. \end{cases}$ 2. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$ 3. $\begin{cases} 2x^2 - 3y = 20, \\ x^2 + 5y = 36. \end{cases}$

4. $\begin{cases} 7x + xy = 20, \\ 2xy + 5x = 22. \end{cases}$ 5. $\begin{cases} 5xy + 3x^2 = 132, \\ 5xy - 3x^2 = 78. \end{cases}$ 6. $\begin{cases} 9x^2 + 5y^2 = 29, \\ 5x^2 - 3y^2 = -7. \end{cases}$

7. $\begin{cases} x^2 + 4x + y^2 + 3y = 30, \\ x^2 + 4x + 6 = y^2 + 3y. \end{cases}$ 8. $\begin{cases} x^2 + 5xy + y^2 = 43, \\ x^2 + 5xy - y^2 = 25. \end{cases}$

9. $\begin{cases} 4x = xy + 5, \\ 7y = xy + 6. \end{cases}$ 10. $\begin{cases} 3x = x^2 + y^2 - 1, \\ 3y = x^2 + y^2 - 7. \end{cases}$

11. $\begin{cases} x + y = 7xy, \\ x - y = 3xy. \end{cases}$ 12. $\begin{cases} 8xy = 5x - 7y - 1, \\ 2xy = 3x + 5y - 9. \end{cases}$

13. $\begin{cases} \dfrac{x+y}{x-y} + 3x = 2\frac{2}{3}, \\ 5\dfrac{x+y}{x-y} - 7x = -8\frac{2}{3}. \end{cases}$ 14. $\begin{cases} 3x + \sqrt{\dfrac{x}{y}} = 30, \\ 5x - 2\sqrt{\dfrac{x}{y}} = 39. \end{cases}$

3. Method of Factoring.—The method of this article depends upon the following principle:

A system of two integral equations

$$\left.\begin{array}{l} P \times Q = 0, \\ R \times S = 0, \end{array}\right\} \text{(I.)}$$

whose first members (when all terms are brought to these members) can be resolved into factors, is equivalent to the four systems,

$$\left.\begin{array}{l} P = 0, \\ R = 0, \end{array}\right\} (a), \quad \left.\begin{array}{l} P = 0, \\ S = 0, \end{array}\right\} (b), \quad \left.\begin{array}{l} Q = 0, \\ R = 0, \end{array}\right\} (c), \quad \left.\begin{array}{l} Q = 0, \\ S = 0, \end{array}\right\} (d),$$

obtained by taking each factor of one equation with each factor of the other.

For, every solution of the given system must reduce either P or Q, or both P and Q, to 0, and at the same time must reduce either R or S, or both R and S, to 0.

Now any solution of (I.) which reduces P to 0 and R to 0, is a solution of (a); any solution which reduces P to 0 and S to 0, is a solution of (b); and so on. Therefore, every solution of the given system is a solution of at least one of the derived systems.

And any solution of (a) reduces P to 0 and R to 0, and therefore reduces $P \times Q$ to 0 and $R \times S$ to 0. Therefore, every solution of (a) is a solution of (I.).

In like manner it can be shown that every solution of the three other derived systems is a solution of the given system.

Ex. 1. Solve the system $(x-2y)(x-3y)=0,$

$$(x+y-4)(x-y+2)=0.$$

The given system is equivalent to the four systems

$$\left.\begin{aligned} x-2y&=0,\\ x+y-4&=0, \end{aligned}\right\}(a), \qquad \left.\begin{aligned} x-2y&=0,\\ x-y+2&=0, \end{aligned}\right\}(b),$$

$$\left.\begin{aligned} x-3y&=0,\\ x+y-4&=0, \end{aligned}\right\}(c), \qquad \left.\begin{aligned} x-3y&=0,\\ x-y+2&=0, \end{aligned}\right\}(d).$$

The solution of (a) is $\frac{8}{3}, \frac{4}{3}$; the solution of (b) is $-4, -2$; the solution of (c) is $3, 1$; the solution of (d) is $-3, -1$.

These are therefore the solutions of the given equations.

Ex. 2. Solve the system $2x^2-7xy+6y^2=0, \qquad (1)$

$$x^2+y^2=13. \qquad (2)$$

The first member of (1) is $(x-2y)(2x-3y)$, and the first member of (2), when 13 is transferred to that member, cannot be resolved into rational factors. The given system is therefore equivalent to the two systems

$$\left.\begin{aligned} x-2y&=0,\\ x^2+y^2&=13, \end{aligned}\right\}(a), \qquad \left.\begin{aligned} 2x-3y&=0,\\ x^2+y^2&=13, \end{aligned}\right\}(b).$$

The solutions of (a) and (b), and therefore of the given system, are respectively

$$2\sqrt{\tfrac{13}{5}}, \sqrt{\tfrac{13}{5}};\ -2\sqrt{\tfrac{13}{5}}, -\sqrt{\tfrac{13}{5}};\ 3, 2;\ -3, -2.$$

4. When all the terms which contain the unknown numbers in both equations of the system are of the second degree, a system can always be derived whose solution is obtained by the method of the preceding article.

Ex. Solve the system $x^2 + xy + 2y^2 = 74,$ (1)

$2x^2 + 2xy + y^2 = 73.$ (2)

Multiplying (1) by 73, $73x^2 + 73xy + 146y^2 = 74 \times 73.$ (3)

Multiplying (2) by 74, $148x^2 + 148xy + 74y^2 = 74 \times 73.$ (4)

Subtracting (3) from (4), $75x^2 + 75xy - 72y^2 = 0,$

or $25x^2 + 25xy - 24y^2 = 0,$

or $(5x - 3y)(5x + 8y) = 0.$

Therefore the given system is equivalent to

$$\left.\begin{aligned} 5x - 3y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\} (a), \qquad \left.\begin{aligned} 5x + 8y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\} (b).$$

The solutions of these systems, and hence of the given system, are respectively 3, 5; −3, −5; 8, −5; −8, 5.

In applying this method to such systems, we must first derive from the given equations a homogeneous equation in which there is no term free from the unknown numbers.

5. Such examples can also be solved by a special device.

Ex. Solve the system $x^2 + 4y^2 = 13,$ (1)

$xy + 2y^2 = 5.$ (2)

In both equations, let $y = tx.$ (3)

Then from (1), $x^2 + 4x^2t^2 = 13,$ whence $x^2 = \dfrac{13}{1 + 4t^2};$ (4)

and from (2), $x^2t + 2x^2t^2 = 5,$ whence $x^2 = \dfrac{5}{t + 2t^2}.$ (5)

Equating values of x^2, $\dfrac{13}{1 + 4t^2} = \dfrac{5}{t + 2t^2}.$ (6)

Whence $t = \frac{1}{3}$, and $t = -\frac{5}{2}.$

When $t = \frac{1}{3}$, $x^2 = \dfrac{13}{1 + 4t^2} = 9$, whence $x = \pm 3.$

When $t = -\frac{5}{2}$, $x^2 = \frac{1}{2}$, whence $x = \pm\sqrt{\frac{1}{2}}.$

When $x = \pm 3$, $y = tx = \frac{1}{3}(\pm 3) = \pm 1.$

When $x = \pm\sqrt{\frac{1}{2}}$, $y = -\frac{5}{2}(\pm\sqrt{\frac{1}{2}}) = \mp\frac{5}{2}\sqrt{\frac{1}{2}}.$

After assuming $y = tx$, we have a system of three equations in x, y, and t. Then the system (1), (2), (3) is equivalent to (3), (4), (5), which is in turn equivalent to (3), (4), and (6). From (6) we obtain the values of t, from (4) the corresponding values of x, and from (3) the corresponding values of y.

The solutions of the given system therefore are

$$3, 1;\ -3, -1;\ \sqrt{\tfrac{1}{2}}, -\tfrac{5}{2}\sqrt{\tfrac{1}{2}};\ -\sqrt{\tfrac{1}{2}}, \tfrac{5}{2}\sqrt{\tfrac{1}{2}}.$$

EXERCISES III.

Solve each of the following systems:

1. $\begin{cases} (x-8)(y-6)=0, \\ x+y=13. \end{cases}$
2. $\begin{cases} (x-5)(y-3)=0, \\ (x-4)(y-7)=0. \end{cases}$
3. $\begin{cases} x^2+xy=78, \\ y^2-xy=7. \end{cases}$
4. $\begin{cases} x^2+4y^2=13, \\ xy+2y^2=5. \end{cases}$
5. $\begin{cases} x^2+xy+y^2=52, \\ xy-x^2=8. \end{cases}$
6. $\begin{cases} x^2-xy+y^2=21, \\ y^2-2xy+15=0. \end{cases}$
7. $\begin{cases} x^2+xy+4y^2=6, \\ 3x^2+8y^2=14. \end{cases}$
8. $\begin{cases} x^2-2xy+3y^2=9, \\ x^2-4xy+5y^2=5. \end{cases}$
9. $\begin{cases} x^2+xy+y^2=13x, \\ x^2-xy+y^2=7x. \end{cases}$
10. $\begin{cases} x^2+y^2=61-3xy, \\ x^2-y^2=31-2xy. \end{cases}$
11. $\begin{cases} x+4\sqrt{(xy)}+4y+\sqrt{x}+2\sqrt{y}=12, \\ \sqrt{x}-2\sqrt{y}=-1. \end{cases}$
12. $\begin{cases} (x-3)(y-2)=0, \\ \dfrac{7}{x}+\dfrac{3}{y}=2. \end{cases}$
13. $\begin{cases} \dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}=0, \\ x^2+2xy=12. \end{cases}$

6. *If the members of one equation of a system of two equations*

$$PR=QS,\ (1) \qquad P=Q,\ (2) \qquad \text{(I.)}$$

contain as factors the corresponding members of the second equation, then the system is equivalent to the following two systems:

$$R=S,\ P=Q,\ (a),\ \text{and}\ P=0,\ Q=0,\ (b).$$

The first equation of the system (a) is obtained by dividing the members of equation (1) of the given system by the corresponding members of equation (2), and the second equation is equation (2) of the given system.

The system (*b*) is obtained by equating to 0 the two members of equation (2) of the given system.

The given system is equivalent to the system

$$PR - QS = 0,\ P - Q = 0, \qquad \text{(II.)}$$

or, replacing Q by P, to the system

$$P(R - S) = 0,\ P - Q = 0. \qquad \text{(III.)}$$

The system (III.) is, by Art. 3, equivalent to the two systems

$$R - S = 0,\ P - Q = 0, \text{ and } P = 0,\ P - Q = 0.$$

That is, to the systems (*a*) and (*b*).

Ex. 1. Solve the system $(x - 1)(x - y + 2) = (y + 1)(x + y)$,
$x - y + 2 = x + y$.

The given system is equivalent to the following two systems:

$$\left.\begin{aligned} x - 1 &= y + 1, \\ x - y + 2 &= x + y, \end{aligned}\right\} (a), \quad \text{and} \quad \left.\begin{aligned} x - y + 2 &= 0, \\ x + y &= 0, \end{aligned}\right\} (b).$$

The solution of (*a*) is 3, 1; the solution of (*b*) is -1, 1.

Ex. 2. Solve the system

$$\begin{aligned} x^2 - y^2 &= 8, \\ x + y &= 3. \end{aligned}$$

The given system is equivalent to the two systems

$$\left.\begin{aligned} x - y &= \tfrac{8}{3}, \\ x + y &= 3, \end{aligned}\right\} (a), \quad \text{and} \quad \left.\begin{aligned} x + y &= 0, \\ 3 &= 0, \end{aligned}\right\} (b).$$

The solution of system (*a*) is $\frac{17}{6}$, $\frac{1}{6}$. Since the equation $3 = 0$ is impossible for finite values of x, the system (*b*) is impossible.

EXERCISES IV.

Solve each of the following systems:

1. $\begin{cases} x^2 - 4y^2 = 21, \\ x - 2y = 3. \end{cases}$

2. $\begin{cases} x^2 - y^2 + (x + y)^2 = 24, \\ x + y = 4. \end{cases}$

3. $\begin{cases} 15x^2 + 2xy - y^2 = 15, \\ 5x - y = 3. \end{cases}$

4. $\begin{cases} (x^2 - 1)(y^2 - 1) = 2800, \\ (x - 1)(y - 1) = 40. \end{cases}$

5. $\begin{cases} (3x - 1)^2 - (4y + 2)^2 = 60, \\ 3x + 4y = 5. \end{cases}$

7. Symmetrical Equations. — A **Symmetrical Equation** is one which remains the same when the unknown numbers are interchanged.

A system of two symmetrical equations can be solved by first finding the values of $x + y$ and $x - y$.

Ex. 1. Solve the system $x^2 + y^2 = 13$, (1)

$xy = 6$. (2)

Multiplying (2) by 2, $2xy = 12$. (3)

Adding (3) to (1), $x^2 + 2xy + y^2 = 25$. (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 1$. (5)

Equating square roots of (4), $x + y = \pm 5$. (6)

Equating square roots of (5), $x - y = \pm 1$. (7)

The given system is equivalent to the system (4), (5), which is equivalent to the systems (6), (7).

The latter systems are

$$\left.\begin{matrix} x+y=5, \\ x-y=1, \end{matrix}\right\} (a), \quad \left.\begin{matrix} x+y=5, \\ x-y=-1, \end{matrix}\right\} (b), \quad \left.\begin{matrix} x+y=-5, \\ x-y=+1, \end{matrix}\right\} (c), \quad \left.\begin{matrix} x+y=-5, \\ x-y=-1, \end{matrix}\right\} (d).$$

The solutions of these four systems are respectively 3, 2; 2, 3; −2, −3; −3, −2.

The solutions of (6) and (7) should be obtained mentally, without writing the equivalent systems (*a*), (*b*), (*c*), (*d*). Each sign of the second member of (6) should be taken in turn with each sign of the second member of (7).

Notice that these solutions differ only in having the values or x and y interchanged. This we should expect from the definition of symmetrical equations.

When the equations are symmetrical, except for sign, the solution can be obtained by a similar method.

Ex. 2. Solve the system $x - y = 1$, (1)

$xy = 2$. (2)

Squaring (1), $x^2 - 2xy + y^2 = 1$. (3)

Adding four times (2) to (3), $x^2 + 2xy + y^2 = 9$. (4)

Equating square roots of (4), $x + y = \pm 3$. (5)

The solutions of (5) and (1) are 2, 1, and −1, −2.

Notice that the solutions in this case differ not only in having the values of x and y interchanged, but also in sign.

Observe that the system (2), (3), and therefore the equivalent system (3), (4), is equivalent to the two systems:

$$\left.\begin{aligned} x - y &= 1, \\ xy &= 2, \end{aligned}\right\} \text{the given system, and} \left\{\begin{aligned} x - y &= -1, \\ xy &= 2. \end{aligned}\right.$$

Consequently, if in the system (3), (4), equation (3) be replaced by (1), the resulting system

$$x - y = 1, \qquad (1)$$

$$x^2 + 2xy + y^2 = 9, \qquad (4)$$

is equivalent to the given system.

8. Many systems which are not symmetrical can be solved by the method of the preceding article.

Ex. 1. Solve the system $2x + 3y = 8, \qquad (1)$

$$xy = 2. \qquad (2)$$

We should first obtain the value of $2x - 3y$.

Squaring (1), $4x^2 + 12xy + 9y^2 = 64. \qquad (3)$

Subtracting 24 times (2) from (3), $4x^2 - 12xy + 9y^2 = 16. \qquad (4)$

Equating square roots of (4), $2x - 3y = \pm 4. \qquad (5)$

The solutions of (1) and (5) are 3, $\frac{2}{3}$; 1, 2.

EXERCISES V.

Solve each of the following systems:

1. $\begin{cases} x + y = 12, \\ xy = 32. \end{cases}$

2. $\begin{cases} x + y = a, \\ xy = b. \end{cases}$

3. $\begin{cases} \frac{1}{2}x + 5y = 37, \\ xy = 28. \end{cases}$

4. $\begin{cases} x - y = 8, \\ xy = -15. \end{cases}$

5. $\begin{cases} x - y = m, \\ xy = n. \end{cases}$

6. $\begin{cases} 6x - 7y = 58, \\ 3xy = -60. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 40, \\ xy = 12. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 181, \\ xy = -90. \end{cases}$

9. $\begin{cases} 25x^2 + 9y^2 = 148, \\ 5xy = 8. \end{cases}$

10. $\begin{cases} 9x^2 + y^2 = 37a^2, \\ xy = -2a^2. \end{cases}$

11. $\begin{cases} 5x^2 + 2y^2 = 5a^2 + 8b^2, \\ xy = 2ab. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 137, \\ x + y = 15. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 61, \\ x + y = 11. \end{cases}$

14. $\begin{cases} 5x + 3y = 11, \\ 25x^2 + 9y^2 = 73. \end{cases}$

15. $\begin{cases} x^2 - y^2 = 28, \\ xy = 48. \end{cases}$

16. $\begin{cases} x^2 - 4y^2 = -3, \\ xy = -1. \end{cases}$

17. $\begin{cases} x^2 + y^2 = 53, \\ x - y = 5. \end{cases}$

18. $\begin{cases} x^2 + y^2 = 74, \\ x - y = 2. \end{cases}$ **19.** $\begin{cases} 9x^2 + y^2 = 82, \\ 3x - y = 10. \end{cases}$ **20.** $\begin{cases} 16x^2 + 49y^2 = 113, \\ 4x + 7y = 1. \end{cases}$

21. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 3, \\ \frac{1}{xy} = 2. \end{cases}$ **22.** $\begin{cases} xy = 80, \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{5}. \end{cases}$ **23.** $\begin{cases} x + y = 16, \\ \frac{1}{x} + \frac{1}{y} = \frac{1}{3}. \end{cases}$

24. $\begin{cases} \frac{x}{y} - \frac{y}{x} = \frac{16}{15}, \\ x - y = 2. \end{cases}$ **25.** $\begin{cases} \frac{1}{x} + \frac{1}{y} = 10, \\ \frac{1}{x^2} + \frac{1}{y^2} = 58. \end{cases}$ **26.** $\begin{cases} x^2 + y^2 = 2\frac{1}{2}xy, \\ \frac{1}{x} + \frac{1}{y} = 1\frac{1}{2}. \end{cases}$

27. $\begin{cases} \sqrt{x} + \sqrt{y} = 5, \\ xy = 36. \end{cases}$ **28.** $\begin{cases} \sqrt{x} + \sqrt{y} = 7, \\ x + y = 37. \end{cases}$ **29.** $\begin{cases} x^2 + x + y = 18 - y^2, \\ xy = 6. \end{cases}$

30. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$ **31.** $\begin{cases} x^2 + y^2 + 7xy = 171, \\ xy = 2(x + y). \end{cases}$

32. $\begin{cases} x^2 + y^2 - (x - y) = 20, \\ xy + x - y = 1. \end{cases}$ **33.** $\begin{cases} x^2 + y^2 - x - y = 22, \\ x + y + xy = -1. \end{cases}$

34. $\begin{cases} x^2 + y^2 + x - y = a, \\ xy + x - y = b. \end{cases}$ **35.** $\begin{cases} x + y = 2, \\ x^2 + y^2 + xy = 3. \end{cases}$

36. $\begin{cases} x + y = 9, \\ x^2 + y^2 - xy = 21. \end{cases}$ **37.** $\begin{cases} x^2 + xy + y^2 = 2m, \\ x^2 - xy + y^2 = 2n. \end{cases}$

38. $\begin{cases} 4xy = 96 - x^2y^2, \\ x + y = 6. \end{cases}$ **39.** $\begin{cases} \sqrt{[(2 + x)(1 + y)]} = 2, \\ \sqrt{(2 + x)} - \sqrt{(1 + y)} = \frac{1}{6}. \end{cases}$

40. $\begin{cases} x + \sqrt{x} + y + \sqrt{y} = 18, \\ (x + \sqrt{x})(y + \sqrt{y}) = 72. \end{cases}$

Simultaneous Quadratic Equations in Three Unknown Numbers.

9. No definite methods can be given for solving simultaneous quadratic equations in three unknown numbers.

Ex. **1.** Solve the system

$$xy = 2, \quad (1) \qquad xz = 3, \quad (2) \qquad yz = 6. \quad (3)$$

Multiplying corresponding members of (1), (2), and (3),

$$x^2y^2z^2 = 36. \tag{4}$$

Equating square roots of (4), $xyz = \pm 6$. (5)

Dividing (5) in turn by (1), (2), and (3),

$$z = \pm 3, \quad y = \pm 2, \quad x = \pm 1.$$

The required solutions are therefore, 1, 2, 3, and -1, -2, -3.

Ex. 2. Solve the system

$$x(y+z)=5, \quad (1) \qquad y(x+z)=8, \quad (2) \qquad z(x+y)=9. \quad (3)$$

Adding corresponding members of (1), (2), and (3), and dividing by 2,

$$xy+xz+yz=11. \qquad (4)$$

Subtracting in turn (1), (2), and (3) from (4),

$$yz=6, \quad (5) \qquad xz=3, \quad (6) \qquad xy=2. \quad (7)$$

The solutions of equations (5), (6), and (7) are 1, 2, 3, and -1, -2, -3.

EXERCISES VI.

Solve each of the following systems:

1. $\begin{cases} xy=30, \\ yz=-60, \\ xz=-50. \end{cases}$

2. $\begin{cases} x\sqrt{y}=a, \\ x\sqrt{z}=b, \\ y\sqrt{z}=c. \end{cases}$

3. $\begin{cases} x^2+y^2=13, \\ x^2+z^2=34, \\ y^2+z^2=29. \end{cases}$

4. $\begin{cases} x^2+yz=5, \\ y^2+xz=5, \\ z^2+xy=5. \end{cases}$

5. $\begin{cases} x^2+y^2=a^2, \\ y^2+z^2=b^2, \\ z^2+x^2=c^2. \end{cases}$

6. $\begin{cases} x(y+z)=5, \\ y(x+z)=8, \\ z(x+y)=9. \end{cases}$

7. $\begin{cases} x^2+xy+y^2=61, \\ x^2+xz+z^2=21, \\ y^2+yz+z^2=13. \end{cases}$

8. $\begin{cases} x(x+y+z)=6, \\ y(x+y+z)=12, \\ z(x+y+z)=18. \end{cases}$

9. $\begin{cases} \dfrac{xyz}{x+y}=2, \\ \dfrac{xyz}{x+z}=\dfrac{3}{2}, \\ \dfrac{xyz}{y+z}=\dfrac{6}{5}. \end{cases}$

10. $\begin{cases} \dfrac{x+y}{xyz}=-(a-b)^2, \\ \dfrac{x+z}{xyz}=-(a-c)^2, \\ \dfrac{y+z}{xyz}=-(b-c)^2. \end{cases}$

11. $\begin{cases} 3x=5y, \\ x(z+2)=yz+32, \\ x(z-1)=(y+1)z-1. \end{cases}$

12. $\begin{cases} x+y+z=a, \\ x(x+y)=b^2, \\ z(z+y)=c^2. \end{cases}$

13. $\begin{cases} xz=y^2, \\ x+y+z=19, \\ x^2+y^2+z^2=133. \end{cases}$

14. $\begin{cases} \sqrt{(x^2+y^2+z^2)}=13, \\ x+y+z=19, \\ x(y+z)=48. \end{cases}$

15. $\begin{cases} xz=360, \\ y(z-10)=40, \\ x(z+8)=400+y(z-2). \end{cases}$

16. $\begin{cases} y=\frac{1}{2}(x+z), \\ x^2+y^2=458, \\ y^2+z^2=730. \end{cases}$

17. $\begin{cases} x^2+y^2+z^2=29, \\ xy+xz+yz=-10, \\ x+y-z=-5. \end{cases}$

18. $\begin{cases} (y+z)(x+y+z)=6, \\ (z+x)(x+y+z)=8, \\ (x+y)(x+y+z)=-6. \end{cases}$

§ 2. SIMULTANEOUS HIGHER EQUATIONS.

1. The solutions of certain equations of higher degree than the second can be made to depend upon the solutions of quadratic equations.

Ex. 1. Solve the system $x^3 + y^3 = 9,$ (1)

$$x + y = 3. \quad (2)$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

Subtracting (3) from the square of (2),

$$3\,xy = 6, \text{ or } xy = 2. \quad (4)$$

The solutions of (2) and (4), and therefore of the given system, are 1, 2, and 2, 1.

Ex. 2. Solve the system $x^4 + y^4 = 17,$ (1)

$$x + y = 3. \quad (2)$$

We first find the value of xy.

Let $xy = z.$ (3)

Squaring (2), $x^2 + 2\,xy + y^2 = 9,$ (4)

or $x^2 + y^2 = 9 - 2\,z.$ (5)

Squaring (5), $x^4 + 2\,x^2y^2 + y^4 = 81 - 36\,z + 4\,z^2,$ (6)

or $x^4 + y^4 = 81 - 36\,z + 2\,z^2.$ (7)

Since $x^4 + y^4 = 17$, we have from (7),

$$2\,z^2 - 36\,z + 81 = 17. \quad (8)$$

Whence $z = 16$, and 2. (9)

Therefore, from (3) and (9), $xy = 16,$ (10)

and $xy = 2.$ (11)

The solutions of (2) and (10) and of (2) and (11) are readily found, and should be checked by substitution.

Ex. 3. Solve the system

$$(x^2 + y^2)(x^3 + y^3) = 45, \quad (1)$$

$$x + y = 3. \quad (2)$$

From (2) $x^2 + y^2 = 9 - 2\,xy,$ (3)

and
$$\begin{aligned} x^3 + y^3 &= 27 - 3\,xy(x + y) \\ &= 27 - 9\,xy, \text{ since } x + y = 3. \quad (4) \end{aligned}$$

Substituting in (1) for x^2+y^2 and x^3+y^3 their values from (3) and (4),

$$(9 - 2\,xy)(27 - 9\,xy) = 45,$$

or $2\,x^2y^2 - 15\,xy = -22.$ (5)

Equation (5) can be solved as a quadratic in xy, and the results combined with equation (2). The results should be checked by substitution.

EXERCISES VII.

Solve each of the following systems:

1. $\begin{cases} x + y = 5, \\ x^3 + y^3 = 35. \end{cases}$

2. $\begin{cases} x - y = 1, \\ x^3 - y^3 = 7. \end{cases}$

3. $\begin{cases} 2(x + y) = 5, \\ 32(x^3 + y^3) = 2285. \end{cases}$

4. $\begin{cases} (x - 1)^3 + (y - 2)^3 = 28, \\ x + y = 7. \end{cases}$

5. $\begin{cases} (x - 7)^3 + (5 - y)^3 = 9, \\ x - y = 5. \end{cases}$

6. $\begin{cases} x^4 - y^4 = 554, \\ x^2 + y^2 = 34. \end{cases}$

7. $\begin{cases} x^4 + y^4 = 82, \\ xy = 3. \end{cases}$

8. $\begin{cases} x^4 + y^4 = 97, \\ x + y = 5. \end{cases}$

9. $\begin{cases} x^4 + y^4 = 257, \\ x - y = 3. \end{cases}$

10. $\begin{cases} (x - 7)^4 + (y - 3)^4 = 257, \\ x - y + 1 = 0. \end{cases}$

11. $\begin{cases} (x^2 - y^2)(x + y) = 9, \\ xy(x + y) = 6. \end{cases}$

12. $\begin{cases} (x + y)(x^2 + y^2) = 175, \\ (x - y)(x^2 - y^2) = 7. \end{cases}$

13. $\begin{cases} x - y = 342, \\ \sqrt[3]{x} - \sqrt[3]{y} = 6. \end{cases}$

14. $\begin{cases} x + y = 30, \\ \sqrt[3]{(x + 7)} + \sqrt[3]{(y - 9)} = 4. \end{cases}$

15. $\begin{cases} x + y + \sqrt{(x + y)} = 12, \\ x^3 + y^3 = 189. \end{cases}$

16. $\begin{cases} \sqrt[4]{(x + 7)} + \sqrt[4]{(y - 5)} = 3, \\ x + y = 15. \end{cases}$

17. $\begin{cases} \sqrt[4]{(90 - x)} - \sqrt[4]{(9 - y)} = 2, \\ x + y = 17. \end{cases}$

18. $\begin{cases} x^4 + y^4 = 4097, \\ x + y + \sqrt{(x + y)} = 12. \end{cases}$

EXERCISES VIII.

MISCELLANEOUS EXAMPLES.

Solve each of the following systems by the methods given in this chapter, or by special devices:

1. $\begin{cases} x - 2y = 2, \\ xy = 12. \end{cases}$

2. $\begin{cases} x(7x - 8y) = 159, \\ 5x + 2y = 7. \end{cases}$

3. $\begin{cases} x + y = x^2, \\ 3y - x = y^2. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 8(x - y), \\ x^2 + y^2 = 50. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 2(x + y), \\ x^2 + y^2 = 100. \end{cases}$

6. $\begin{cases} \dfrac{x + y}{x - y} + \dfrac{x - y}{x + y} = 5\frac{1}{5}, \\ 2x^2 - 3y^2 = 24. \end{cases}$

7. $\begin{cases} \dfrac{x^2 - 2xy + 3y^2}{3x^2 - 2xy + y^2} = \dfrac{1}{3}, \\ x^2 - 3y = 1. \end{cases}$

8. $\begin{cases} (5x - 3)(3y + 2) = 0, \\ (4x + 5)(2y - 3) = 0. \end{cases}$

9. $\begin{cases} (2x+3y-7)(x-4y+2) = 0, \\ x^2 + y^2 + 2x - 7y = 2. \end{cases}$

10. $\begin{cases} x - y = 5(x^2 - y^2), \\ 2x^2 + 3xy + 4x + 5y = 0. \end{cases}$

11. $\begin{cases} x^2 + y^2 + 5x - 9y = 84, \\ x^2 - y^2 + 5x + 9y = 84. \end{cases}$

12. $\begin{cases} 3x^2 + 5y^2 + 4x + 3y = 9, \\ 3x^2 + 5y^2 + 2x - 4y = 14. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 485, \\ x^2y^2 = 57834 - 5xy. \end{cases}$

14. $\begin{cases} x^2 + y^2 + x - y = 62, \\ (x^2 + y^2)(x - y) = 61. \end{cases}$

15. $\begin{cases} 3(x + y)^2 = \frac{8}{7}(x + y) + \frac{1280}{7}, \\ \frac{1}{5}x^2y^2 - \frac{2}{15}xy = 43. \end{cases}$

16. $\begin{cases} \sqrt{(3x+3y-5)} + 16 - 6y = 6x, \\ 2x^2y^2 + 2 = 5xy. \end{cases}$

17. $\begin{cases} 2x + 3y + 6xy = 11, \\ 4x^2 + 9y^2 + 12xy = x^2y^2 - 11. \end{cases}$

18. $\begin{cases} \left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right) = -\frac{9}{4}, \\ \left(x - \frac{1}{x}\right)^2 + \left(y - \frac{1}{y}\right)^2 = \frac{9}{2}. \end{cases}$

19. $\begin{cases} x^2 + y^2 = \frac{40}{x + y}, \\ xy = \frac{12}{x + y}. \end{cases}$

20. $\begin{cases} x^3 - y^3 = 26, \\ x - y = 2. \end{cases}$

21. $\begin{cases} x^3y^2 - x^2y^3 = 1152, \\ x^2y - xy^2 = 48. \end{cases}$

22. $\begin{cases} x + y = 19, \\ \sqrt[3]{x} + \sqrt[3]{y} = 4. \end{cases}$

23. $\begin{cases} x - y - \sqrt{(x - y)} = 2, \\ x^3 - y^3 = 2044. \end{cases}$

24. $\begin{cases} \frac{xy}{x + y} = a, \\ \frac{x^2y^2}{x^2 + y^2} = b^2. \end{cases}$

25. $\begin{cases} xy + \frac{x}{y} = \frac{5}{3}, \\ \frac{1}{xy} + \frac{y}{x} = \frac{20}{3}. \end{cases}$

26. $\begin{cases} x^n + y^n = a^n, \\ xy = b. \end{cases}$

27. $\begin{cases} x^4 + y^4 = 14x^2y^2, \\ x + y = a. \end{cases}$

28. $\begin{cases} 3y(x^2 + y^2) = 10x, \\ 2x(x^2 - y^2) = 48y. \end{cases}$

29. $\begin{cases} (x^2 + y^2)(x^3 + y^3) = 455, \\ x + y = 5. \end{cases}$

30. $\begin{cases} x^4 + x^2y^2 + y^4 = 133, \\ x^2 - xy + y^2 = 7. \end{cases}$

31. $\begin{cases} x^4 + x^2y^2 + y^4 = 84x^2, \\ x^2 + xy + y^2 = 14x. \end{cases}$

32. $\begin{cases} x^4 - x^2 + y^4 - y^2 = 84, \\ x^2 + xy + y^2 = 19. \end{cases}$

33. $\begin{cases} x^2 - x^2y^2 + y^2 + 23 = 0, \\ x - xy + y + 1 = 0. \end{cases}$

34. $\begin{cases} x^3 + y^3 + xy(x + y) = 65, \\ x^2y^2(x^2 + y^2) = 468. \end{cases}$

35. $\begin{cases} (x + y)(xy + 1) = 18xy, \\ (x^2+y^2)(x^2y^2+1) = 320x^2y^2. \end{cases}$

36. $\begin{cases} x + y + \sqrt{(xy)} = 14, \\ x^2 + y^2 + xy = 84. \end{cases}$

37. $\begin{cases} x + y - \sqrt{(xy)} = 7, \\ x^2 + y^2 + xy = 133. \end{cases}$

38. $\begin{cases} y^2 = xz, \\ x + y + z = 28, \\ xyz = 512. \end{cases}$

39. $\begin{cases} x + y + z = 5, \\ x^2 + y^2 = z^2, \\ x^3 + y^3 + z^3 = 8. \end{cases}$

40. $\begin{cases} xu = yz, \\ x + u = 18, \\ y + z = 14, \\ x^2 + y^2 + z^2 + u^2 = 340. \end{cases}$

41. $\begin{cases} xu = yz, \\ x + u = 7, \\ y + z = 5, \\ x^4 + y^4 + z^4 + u^4 = 1394. \end{cases}$

§ 3. PROBLEMS.

1. Pr. The front wheel of a carriage makes 6 more revolutions than the hind wheel in traveling 360 feet. But if the circumference of each wheel were 3 feet greater, the front wheel would make only 4 revolutions more than the hind wheel in traveling the same distance as before. What are the circumferences of the two wheels?

Let x stand for the number of feet in the circumference of front wheel, and y for the number of feet in the circumference of hind wheel. Then in traveling 360 feet the front wheel makes $\frac{360}{x}$ revolutions, and the hind wheel makes $\frac{360}{y}$ revolutions.

By the first condition, $$\frac{360}{x} = \frac{360}{y} + 6. \tag{1}$$

If 3 feet were added to the circumference of each wheel, the front wheel would make $\frac{360}{x+3}$ revolutions, and the hind wheel $\frac{360}{y+3}$ revolutions.

By the second condition, $$\frac{360}{x+3} = \frac{360}{y+3} + 4. \tag{2}$$

Whence $x = 12$, the circumference of the front wheel, and $y = 15$, the circumference of the hind wheel.

EXERCISES IX.

1. The square of one number increased by ten times a second number is 84, and is equal to the square of the second number increased by ten times the first.

2. The sum of two numbers is 20, and the sum of the square of the one diminished by 13 and the square of the other increased by 13 is 272. What are the numbers?

3. Find two numbers such that their difference added to the difference of their squares shall be 150, and their sum added to the sum of their squares shall be 330.

4. Find two numbers whose sum is equal to their product and also to the difference of their squares.

5. The sum of the fourth powers of two numbers is 1921, and the sum of their squares is 61. What are the numbers?

6. If a number of two digits be multiplied by its tens' digit, the product will be 390. If the digits be interchanged and the resulting number be multiplied by its tens' digit, the product will be 280. What is the number?

7. If a number of two digits be divided by the product of its digits, the quotient will be 2. If 27 be added to the number, the sum will be equal to the number obtained by interchanging the digits. What is the number?

8. The product of the two digits of a number is equal to one-half of the number. If the number be subtracted from the number obtained by interchanging the digits, the remainder will be equal to three-halves of the product of the digits of the number. What is the number?

9. If the difference of the squares of two numbers be divided by the first number, the quotient and the remainder will each be 5. If the difference of the squares be divided by the second number, the quotient will be 13 and the remainder 1. What are the numbers?

10. The sum of the three digits of a number is 9. If the digits be written in reverse order, the resulting number will exceed the original number by 396. The square of the middle digit exceeds the product of the first and the third digit by 4. What is the number?

11. A rectangular field is 119 yards long and 19 yards wide. How many yards must be added to its width and how many yards must be taken from its length, in order that its area may remain the same while its perimeter is increased by 24 yards?

12. The floor of a room contains 30⅓ square yards, one wall contains 21 square yards, and an adjacent wall contains 13 square yards. What are the dimensions of the room?

13. A merchant bought a number of pieces of cloth of two different kinds. He bought of each kind as many pieces and paid for each yard half as many dollars as that kind contained yards. He bought altogether 19 pieces and paid for them $921.50. How many pieces of each kind did he buy?

14. The diagonal of a rectangle is 20⅖ feet. If the length of one side be increased by 14 feet and the length of the other side be diminished by 2⅖ feet, the diagonal will be increased by 12⅖ feet. What are the lengths of the sides of the rectangle?

15. A certain number of coins can be arranged in the form of one square, and also in the form of two squares. In the first arrangement each side of the square contains 29 coins, and in the second arrangement one square contains 41 more coins than the other. How many coins are there in a side of each square of the second arrangement?

16. A piece of cloth after being wet shrinks in length by one-eighth and in breadth by one-sixteenth. The piece contains after shrinking 3.68 fewer square yards than before shrinking, and the length and breadth together shrink 1.7 yards. What was the length and breadth of the piece?

17. A merchant paid $125 for two kinds of goods. He sold the one kind for $91 and the other for $36. He thereby gained as much per cent on the first kind as he lost on the second. How much did he pay for each kind?

18. Two workmen can do a piece of work in 6 days. How long will it take each of them to do the work, if it takes one 5 days longer than the other?

19. Two men, A and B, receive different wages. A earns $42, and B $40. If A had received B's wages a day, and B had received A's wages, they would have earned together $4 more. How many days does each work, if A works 8 days more than B, and what wages does each receive?

20. In 8 hours workmen remove a pile of stones from one place to another. Had there been 8 more workmen, and had each one carried 5 pounds less at each trip, they would have completed the work in 7 hours. Had there been 8 fewer workmen and had each one carried 11 pounds more at each trip, they would have completed the work in 9 hours. How many workmen were there and how many pounds did each one carry at every trip?

21. A man has two square fields in which he wishes to plant trees, the outer rows to be distant from the edges by half the distance between the rows. If he plants the trees in the first field 2½ yards apart, and in the second field 2¼ yards apart, he will need 11,412 trees. But if he plants the trees in the first field 2¾ yards apart, and in the second field 3 yards apart, he will need only 569 trees. How long is a side of each of the fields?

22. A tank can be filled by one pipe and emptied by another. If, when the tank is half full of water, both pipes be left open 12 hours, the tank will be emptied. If the pipes be made smaller, so that it will take the one pipe one hour longer to fill the tank and the other one hour longer to empty it, the tank, when half full of water, will then be emptied in 15¾ hours. In what time will the empty tank be filled by the one pipe, and the full tank be emptied by the other?

CHAPTER XXV.

RATIO, PROPORTION, AND VARIATION.

§ 1. RATIO.

1. The **Ratio** of one number to another is the relation between the numbers which is expressed by the quotient of the first divided by the second.

E.g., the ratio of 6 to 4 is expressed by $\frac{6}{4}$, $= \frac{3}{2}$.

The ratio of one number to another is frequently expressed by placing a colon between them; as $5 : 7$.

The first number in a ratio is called the **First Term**, or the **Antecedent** of the ratio, and the second number the **Second Term**, or the **Consequent** of the ratio.

Thus, in the ratio $a : b$, a is the first term, and b the second.

2. Since, by definition, a ratio is a fraction, all the properties of fractions are true of ratios; as $a : b = ma : mb$.

3. The definition given in Art. 1 has reference to the ratio of one *number* to another. But it is frequently necessary to compare concrete quantities, as the length of one line with the length of another line, etc.

If two concrete quantities of the same kind can be expressed by two rational numbers in terms of the same unit, then the ratio of the one quantity to the other is defined as the ratio of the one number to the other.

E.g., the ratio of $2\frac{1}{2}$ yards to $1\frac{1}{7}$ yards is $2\frac{1}{2} : 1\frac{1}{7}$, $= \frac{2\frac{1}{2}}{1\frac{1}{7}} = \frac{35}{16}$.

Observe that by this definition the ratio of two concrete quantities is a number. Also that the quantities to be compared must be of the same kind. Dollars cannot be compared with pounds, etc.

4. If two concrete quantities cannot be expressed by two rational numbers in terms of the same unit, they are said to be **Incommensurable** one to the other.

Thus, if the lengths of the two sides of a right triangle be equal, the length of the hypothenuse cannot be expressed by a rational number in terms of a side as a unit, or any fraction of a side as a unit.

If a side be taken as the unit, the hypothenuse is expressed by $\sqrt{2}$, an irrational number. And the ratio of the hypothenuse to a side is $\sqrt{2} : 1, = \sqrt{2}$, a number comprised in the number system. In the following article we will prove that the ratio of any two incommensurable quantities can be expressed as a number comprised in the number system.

5. Let P and Q be two incommensurable quantities. We assume that the ratio $P : Q$ is greater than the ratio $P' : Q$, wherein P' is less than P and is commensurable with Q, and that the ratio $P : Q$ is less than the ratio $P'' : Q$, wherein P'' is greater than P and is commensurable with Q.

Let us take $\frac{1}{n}Q$ as the unit. Then we can find two consecutive integral multiples of $\frac{1}{n}Q$, which are therefore commensurable with Q, between which P lies. Let $\frac{m}{n}Q$ and $\frac{m+1}{n}Q$ be these multiples. The ratios of these multiples to Q are respectively $\frac{m}{n}$ and $\frac{m+1}{n}$. Then by the hypothesis

$$\frac{m}{n} < P : Q < \frac{m+1}{n}.$$

The two rational numbers $\frac{m}{n}$ and $\frac{m+1}{n}$, between which the ratio $P : Q$ lies, have the properties (i.) and (ii.), Art. 6, Ch. XVIII. They therefore define an irrational number.

EXERCISES I.

What is the ratio of

1. $6a$ to $9b$?
2. $\frac{3}{5}a^2b$ to $\frac{6}{11}ab^2$?
3. $9\frac{1}{2}x^3y$ to $7\frac{3}{5}xy^3$?
4. $\frac{1}{a}$ to $\frac{1}{b}$?
5. $\frac{a}{b}$ to $\frac{c}{d}$?
6. $\frac{a}{x-3}$ to $\frac{1}{(x-3)^2}$?
7. Which is the greater ratio, $a + 2b : a + b$ or $a + 3b : a + 2b$?
8. If $\frac{6x + 2y}{3x - y} = 10$, what is the value of the ratio $x : y$?

§ 2. PROPORTION.

1. A **Proportion** is an equation whose members are two equal ratios.

E.g., $4 : 3 = 8 : 6$, read *the ratio of* 4 *to* 3 *is equal to the ratio of* 8 *to* 6, or 4 *is to* 3 *as* 8 *is to* 6.

Instead of the equality sign a double colon is frequently used; as $4 : 3 :: 8 : 6$.

2. Four numbers are said to be *in proportion*, or to be *proportional*, when the first is to the second as the third is to the fourth.

E.g., the numbers 4, 3, 8, 6 are proportional, since $4 : 3 = 8 : 6$.

The individual numbers are called the **Proportionals**, or **Terms** of the proportion.

The **Extremes** of a proportion are its first and last terms; as 4 and 6 above.

The **Means** of a proportion are its second and third terms; as 3 and 8 above.

The **Antecedents** and **Consequents** of a proportion are the antecedents and consequents of its two ratios.

E.g., 4 and 8 are the antecedents, and 3 and 6 the consequents of the proportion $4 : 3 = 8 : 6$.

Principles of Proportions.

3. *In any proportion the product of the extremes is equal to the product of the means.*

If $a : b = c : d$, we are to prove $ad = bc$.

By § 1, Art. 1, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

4. *If the product of two numbers be equal to the product of two other numbers, the four numbers are in proportion.*

Let $ad = bc$.

Dividing by bd, $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$; (1)

by cd, $\frac{a}{c}=\frac{b}{d}$, or $a:c=b:d$; (2)

by ab, $\frac{d}{b}=\frac{c}{a}$, or $d:b=c:a$; (3)

by ac, $\frac{d}{c}=\frac{b}{a}$, or $d:c=b:a$. (4)

Interchanging the ratios in (1), (2), (3), (4),

$$c:d=a:b; \quad (5)$$

$$b:d=a:c; \quad (6)$$

$$c:a=d:b; \quad (7)$$

$$b:a=d:c. \quad (8)$$

Notice that the two numbers of either product may be taken as the extremes, the other two as the means. In (1) to (4), a and d are the extremes, c and b the means; in (5) to (8), d and a are the means, c and b the extremes.

5. In Art. 4, we may regard the proportions (2) to (8) as being derived from (1), and thus obtain the following properties of a proportion:

(i.) *The means may be interchanged;* as in (2).

(ii.) *The extremes may be interchanged;* as in (3).

(iii.) *The means may be interchanged, and at the same time the extremes;* as in (4).

(iv.) *The means may be taken as the extremes, and the extremes as the means;* as (8) from (1), (7) from (2), etc.

6. *If any three terms of a proportion be given, the remaining term can be found.*

Ex. What is the second term of a proportion, whose first, third, and fourth terms are 10, 16, and 8 respectively?

Letting x stand for the second term, we have

$$10:x=16:8, \text{ or } 16x=80; \text{ whence } x=5.$$

7. *The products, or the quotients, of the corresponding terms of two proportions form again a proportion.*

If $$a:b=c:d,\text{ or }\frac{a}{b}=\frac{c}{d},\qquad(1)$$

and $$x:y=z:u,\text{ or }\frac{x}{y}=\frac{z}{u},\qquad(2)$$

we have, multiplying corresponding members of (1) and (2),

$$\frac{ax}{by}=\frac{cz}{du};\text{ whence }ax:by=cz:du.$$

Dividing the members of (1) by the corresponding members of (2), we have

$$\frac{\frac{a}{x}}{\frac{b}{y}}=\frac{\frac{c}{z}}{\frac{d}{u}};\text{ whence }\frac{a}{x}:\frac{b}{y}=\frac{c}{z}:\frac{d}{u}.$$

8. *In any proportion, the sum of the first two terms is to the first (or the second) term as the sum of the last two terms is to the third (or the fourth) term.*

Let $$a:b=c:d.$$

Then $$\frac{a}{b}=\frac{c}{d}.$$

Adding 1 to both members, $\frac{a}{b}+1=\frac{c}{d}+1,$

or $$\frac{a+b}{b}=\frac{c+d}{d}.$$

Whence $$a+b:b=c+d:d.$$

In like manner it can be proved that

$$a+b:a=c+d:c.$$

These two proportions are said to be derived from the given proportion by **Composition.**

9. *In any proportion, the difference of the first two terms is to the first (or the second) term as the difference of the last two terms is to the third (or the fourth) term.*

If $$a : b = c : d,$$
then $$a - b : a = c - d : c, \text{ and } a - b : b = c - d : d.$$

The proof is similar to that of Art. 8.

These two proportions are said to be derived from the given proportion by **Division.**

10. *In any proportion, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let $$a : b = c : d.$$
By Art. 8, $$a + b : b = c + d : d;$$
and by Art. 9, $$a - b : b = c - d : d.$$
Then by Art. 7, $$\frac{a+b}{a-b} : 1 = \frac{c+d}{c-d} : 1,$$
or $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$
Whence $$a + b : a - b = c + d : c - d.$$

This proportion is said to be derived from the given one by **Composition and Division.**

11. A **Continued Proportion** is one in which the consequent of each ratio is the antecedent of the following ratio; as,

$$a : b = b : c = c : d = \text{etc.}$$

12. In the continued proportion

$$a : b = b : c,$$

b is called a **Mean Proportional** between a and c, and c is called the **Third Proportional** to a and b.

13. *The mean proportional between any two numbers is equal to the square root of their product.*

From $$a : b = b : c,$$
we have, by Art. 3, $b^2 = ac$; whence $b = \sqrt{(ac)}$.

14. The following examples are applications of the preceding theory:

Ex. 1. Find a mean proportional between 5 and 20.

Let x stand for the required proportional.

Then, by Art. 13, $x = \sqrt{(5 \times 20)} = 10.$

Ex. 2. If $$a : b = c : d,$$
then $$ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2.$$

Let $$\frac{a}{b} = \frac{c}{d} = x.$$

Then $$a = bx \text{ and } c = dx.$$

Therefore $$ab + cd = b^2x + d^2x,$$
and $$ab - cd = b^2x - d^2x.$$

We then have
$$\frac{ab + cd}{ab - cd} = \frac{b^2x + d^2x}{b^2x - d^2x} = \frac{b^2 + d^2}{b^2 - d^2}.$$

Whence $$ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2.$$

Ex. 3. Solve the equation
$$\frac{\sqrt{(2 + x)} + \sqrt{(2 - x)}}{\sqrt{(2 + x)} - \sqrt{(2 - x)}} = 2, = \frac{2}{1}.$$

By composition and division,
$$\frac{\sqrt{(2 + x)}}{\sqrt{(2 - x)}} = \frac{3}{1}.$$

Squaring and clearing of fractions,
$$2 + x = 18 - 9x; \text{ whence } x = \tfrac{8}{5}.$$

EXERCISES II.

Verify each of the following proportions:

1. $2\frac{1}{2} : 1\frac{1}{3} = 1\frac{1}{2} : \frac{4}{5}$.
2. $14\frac{2}{3} : 4\frac{2}{5} = 200 : 60$.
3. $\dfrac{4ab}{a^2 - b^2} : \dfrac{a^2 + b^2}{a - b} = \dfrac{2ab}{a^4 - b^4} : \dfrac{1}{2a - 2b}$.

Form proportions from each of the following products, in eight different ways:

4. $2x = 3y$.
5. $m^2 = n^2$.
6. $a^3 - b^3 = x^2 - y^2$.

Find a fourth proportional to

7. 1, 2, and 8.
8. $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$.
9. ab, ac, and b.

Find a third proportional to

10. 2 and 6.
11. $\frac{1}{3}$ and $\frac{1}{6}$.
12. a and b.

Find a mean proportional between

13. 2 and 18.
14. $\frac{1}{3}$ and $\frac{3}{4}$.
15. a^2b and ab^2.
16. $\dfrac{a + b}{a - b}$ and $\dfrac{a^2 - b^2}{a^2b^2}$.
17. $\dfrac{a^2 + 1}{a^2 - 1}$ and $\frac{1}{4}(a^4 - 1)$.

Find a value of x to satisfy each of the following proportions:

18. $x : 2 = 12 : 3.$ **19.** $\frac{1}{2} + \sqrt{a} : \frac{1}{4} - a = x : \sqrt{a} - 2a.$

20. $a + b + \dfrac{2b^2}{a-b} : \dfrac{(a+b)^2}{2ab} - 1 = x : a - b.$

If $a : b = c : d$, prove that

21. $a + c : b + d = a^2d : b^2c.$ **22.** $a^2 + b^2 : a^2 - b^2 = c^2 + d^2 : c^2 - d^2.$

23. $(a \pm b)^2 : ab = (c \pm d)^2 : cd.$

24. $2a + 3b : 4a + 5b = 2c + 3d : 4c + 5d.$

25. $a + b : c + d = \sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)}.$

26. $\sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)} = \sqrt[3]{(a^3 + b^3)} : \sqrt[3]{(c^3 + d^3)} = a : c.$

If b be a mean proportional between a and c, prove that

27. $a^2 \pm b^2 : b^2 \pm c^2 = a : c.$ **28.** $a^2b^2c^2\left(\dfrac{1}{a^3} + \dfrac{1}{b^3} + \dfrac{1}{c^3}\right) = a^3 + b^3 + c^3.$

If $a : b = b : c = c : d$, prove that

29. $(b + c)(b + d) = (a + c)(c + d).$

30. $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$

If $a : b = c : d = e : f$, prove that

31. $a^2 : b^2 = ce : df.$ **32.** $c^2 + d^2 : e^2 + f^2 = cd : ef.$

33. $a^2 + c^2 + e^2 : b^2 + d^2 + f^2 = c^2 : d^2.$

34. If $\dfrac{ay - bx}{c} = \dfrac{cx - az}{b} = \dfrac{bz - cy}{a}$, then $x : a = y : b = z : c.$

Solve each of the following equations:

35. $\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}} = 3.$ **36.** $\dfrac{\sqrt{(a+x)} - \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}} = \dfrac{1}{\sqrt{b}}.$

37. Find two numbers whose ratio is 7 : 5, and the difference of whose squares is 96.

38. A works 6 days with 2 horses, and B works 5 days with 3 horses. What is the ratio of A's work to B's work?

39. The ratio of a father's age to his son's age is 9 : 5. If the father is 28 years older than the son, how old is each?

40. Find three numbers in a continued proportion whose sum is 39, and whose product is 721.

41. Find two numbers such that if 1 be added to the first and 8 to the second, the sums will be in the ratio 1 : 2, and if 1 be subtracted from each number, the remainders will be in the ratio 2 : 3.

42. What is the ratio of the numerator of a fraction to its denominator, if the fraction be unchanged when a is added to its numerator and b to its denominator?

43. The sum of the means of a proportion is 7, the sum of the extremes is 8, and the sum of the squares of all the terms is 65. What is the proportion?

§ 3. VARIATION.

1. Frequently two numbers or quantities are so related to each other that a change in the value of one produces a corresponding change in the value of the other.

Thus, the distance a train runs in one hour depends upon its speed, and increases or decreases when its speed increases or decreases.

The illumination made by a light depends upon the intensity of the light, and varies when the intensity varies.

The value of y given by the equation $y = 2x - 3$ depends upon the value of x and varies when the value of x varies.

Thus, if $x = 1$, $y = -1$; if $x = 2$, $y = 1$, etc.

We shall in this chapter consider only the simplest kinds of variation.

2. Direct Variation. — Two quantities are said to *vary directly*, one as the other, when their ratio is constant.

Thus, if x varies directly as y, then $\frac{x}{y} = k$, a constant.

For example, if a train runs at a uniform speed, the number of miles it runs varies directly as the number of hours. If it runs at the rate of 30 miles an hour, in 1 hour it will run 30 miles, in 2 hours 60 miles, in 3 hours 90 miles, and so on; and the ratios 1 : 30, 2 : 60, 3 : 90, etc., are equal.

The symbol of direct variation, $\propto$, is read *varies directly as*.

The word *directly* is frequently omitted.

If $y = 3x$, then $y \propto x$ (read *y varies as x*), since $\frac{y}{x} = 3$, a constant.

3. Inverse Variation. — One quantity is said to *vary inversely* as another when the first varies as the *reciprocal* of the second.

Thus, if x varies inversely as y, then $x \propto \frac{1}{y}$.

Therefore, $\frac{x}{\frac{1}{y}} = k$, a constant; whence $xy = k$.

That is, if one quantity varies inversely as another, the product of the quantities is constant.

If 6 men can do a piece of work in 12 hours, 3 men can do the same work in 24 hours, and 1 man in 72 hours, and the products 6×12, 3×24, 1×72 are equal. That is, the number of hours varies inversely as the number of men working.

If $y = \frac{3}{x}$, y varies inversely as x, since $xy = 3$.

4. Joint Variation. — One quantity is said to *vary* as two others *jointly*, when it varies as the product of the others.

Thus, if x varies as y and z jointly, then $\frac{x}{yz} = k$, a constant.

For example, the number of miles a train runs varies as the number of hours and the number of miles it runs an hour jointly. It will run 40 miles in 2 hours at a rate of 20 miles an hour, 90 miles in 3 hours at the rate of 30 miles an hour,

and $$\frac{40}{2 \times 20} = \frac{90}{3 \times 30} = \frac{120}{5 \times 24}.$$

5. One quantity is said to vary directly as a second and inversely as a third, when it varies as the second and the reciprocal of the third jointly.

Thus, if x varies directly as y and inversely as z, then $\frac{x}{y \cdot \frac{1}{z}} = k$, a constant; or $\frac{xz}{y} = k$.

6. In all the preceding cases of variation, the constant can be determined when any set of corresponding values of the quantities is known.

Ex. 1. If $x \propto y$, and $x = 3$ when $y = 5$, what is the value of the constant?

We have $\frac{x}{y} = k$, or $x = ky$.

Therefore, when $x = 3$ and $y = 5$,

$3 = 5k$, whence $k = \frac{3}{5}$.

Consequently $x = \frac{3}{5}y$.

Ex. 2. If x varies inversely as y, and if $y = 4$ when $x = 7$, find the value of x when $y = 12$.

From $xy = k$, we obtain $k = 28$.

Therefore $xy = 28$.

Consequently, when $y = 12$, $12x = 28$; whence $x = 2\frac{1}{3}$.

Ex. 3. The volume of a gas varies inversely as the pressure when the temperature is constant. When the pressure is 15, the volume is 20; what is the volume when the pressure is 20?

Let v stand for the volume and p for the pressure.

Then from $pv = k$ we obtain $k = 300$.

Therefore $pv = 300$.

Consequently, when $p = 20$, $20v = 300$; whence $v = 15$.

EXERCISES III.

If $x \propto y$, what is the expression for x in terms of y,

1. If $x = 10$ when $y = \frac{2}{3}$? **2.** If $x = a$ when $y = 2a$?

3. If $x \propto y^2$, and $x = 5$ when $y = 7$, what is the expression for x in terms of y?

4. If $x \propto \sqrt{y}$, and $x = 3(a^3 + b^3)$ when $y = 25(a^2 + 2ab + b^2)$, what is the expression for y in terms of x?

If $x \propto \frac{1}{y}$, what is the expression for x in terms of y,

5. If $x = 10$ when $y = \frac{4}{5}$? **6.** If $x = 3\frac{1}{4}$ when $y = \frac{15}{16}$?

7. If $x \propto \frac{1}{y^2}$, and $x = 4\frac{1}{2}$ when $y = \frac{2}{3}$, what is the expression for y in terms of x?

8. If $x \propto \frac{1}{\sqrt{y}}$, and $x = 4$ when $y = 25$, what is the expression for x in terms of y?

9. If $x \propto y$, and $x = 10$ when $y = 5$, what is the value of x when $y = 12\frac{1}{2}$?

10. If $x \propto y$, and $x = a$ when $y = \frac{3}{4}a^2$, what is the value of y when $x = a^2b$?

11. If $x \propto y^2$, and $x = 5$ when $y = -3$, what is the value of x when $y = 15$?

12. If $x \propto \sqrt{y}$, and $x = a + m$ when $y = (a - m)^2$, what is the value of x when $y = (a + m)^4$?

13. If $x \propto \frac{1}{y}$, and $x = 3$ when $y = \frac{2}{3}$, what is the value of x when $y = 4\frac{1}{7}$?

14. The circumference of a circle whose radius is 6 feet is 37.7 feet. What is the circumference of a circle whose radius is 9.5 feet, if it be known that the circumference varies as the radius?

15. An ox is tied by a rope 20 yards long in the center of a field, and eats all the grass within his reach in $2\frac{1}{2}$ days. How many days would it have taken the ox to eat all the grass within his reach if the rope had been 10 yards longer?

16. The volume of a sphere whose radius is 7 inches is 1437.3 cubic inches. What is the volume of a sphere whose radius is 10 inches, if it be known that the volume varies as the cube of the radius?

It has been found by experiment that the distance a body falls from rest varies as the square of the time.

17. If a body falls 256 feet in 4 seconds, how far will it fall in 10 seconds?

18. From what height must a body fall to reach the earth after 15 seconds?

It has been found by experiment that the velocity acquired by a body falling from rest varies as the time.

19. If the velocity of a falling body is 160 feet after 5 seconds, what will be the velocity after 8 seconds?

20. How long must a body have been falling to have acquired a velocity of 256 feet?

21. The surface of a cube whose edge is 5 inches is 150 square inches. What is the surface of a cube whose edge is 9 inches, if it be known that the surface varies as the square of its edge?

22. It has been found by experiment that the weight of a body varies inversely as the square of its distance from the center of the earth. If a body weighs 30 pounds on the surface of the earth (approximately 4000 miles from the center), what would be its weight at a distance of 24,000 miles from the surface of the earth?

It has been found by experiment that the illumination of an object varies inversely as the square of its distance from the source of light.

23. If the illumination of an object at a distance of 10 feet from a source of light is 2, what is the illumination at a distance of 40 feet?

24. To what distance must an object which is now 10 feet from a source of light be removed in order that it shall receive only one-half as much light?

25. At what distance will a light of intensity 10 give the same illumination as a light of intensity 8 gives at a distance of 50 feet?

CHAPTER XXVI.

DOCTRINE OF EXPONENTS.

1. We have already abbreviated such products as

$$aa,\ aaa,\ aaaa,\ \cdots,\ aaa \cdots n \text{ factors},$$

by a^2, a^3, a^4, $\cdots$, a^n, respectively, and called them the *second*, *third*, *fourth*, $\cdots$, *nth*, powers of a. This definition of the symbol a^n requires the exponent n to be a *positive integer*.

Thus 2^5 means the product of 5 factors, each equal to 2. But 2^0 has, as yet, no meaning, since 2 cannot be taken 0 times as a factor. For a similar reason, 2^{-5}, $2^{\frac{1}{2}}$, $2^{\sqrt{3}}$, and $2^{\sqrt{-1}}$, are, as yet, meaningless.

2. Nevertheless, having introduced into Algebra the symbol a^n, it is natural to inquire what it may mean when n is 0, a *rational negative* or *fractional number*, an *irrational number*, etc.

We shall find that, by enlarging our conception of *powers*, quite clear and definite meanings can be given to such expressions as 2^0, 3^{-2}, $4^{\frac{1}{2}}$, $5^{\sqrt{2}}$, $6^{\sqrt{-1}}$.

3. The discussion of powers, in general, therefore naturally divides itself into six cases.

(1) Powers with *positive integral exponents.*

(2) Powers with *zero exponents.*

(3) Powers with *negative integral exponents.*

(4) Powers with *fractional* (*positive* or *negative*) *exponents.*

(5) Powers with *irrational exponents.*

(6) Powers with *imaginary exponents.*

The consideration of powers with imaginary exponents will be given in Part II., Text-Book of Algebra.

Positive Integral Powers.

4. The principles upon which operations with positive integral powers depend have been proved in the preceding chapters.

For the sake of emphasis, and for convenience of reference in enlarging our conceptions of powers, we restate them here:

(i.) $$a^m a^n = a^{m+n}.$$

(ii.) $$\frac{a^m}{a^n} = a^{m-n}, \text{ when } m > n; \quad \frac{a^m}{a^n} = 1, \text{ when } m = n;$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ when } m < n.$$

(iii.) $(a^m)^n = a^{mn}.$ (iv.) $(ab)^m = a^m b^m.$

(v.) $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Zeroth Powers.

5. The meaning of a symbol may be defined by assuming that it stands for the result of a definite operation, as was done in letting

$$a^n = a \cdot a \cdot a \cdot \cdots n \text{ factors};$$

or by enlarging the meaning of some operation or law which was previously restricted in its application.

In the latter way, negative numbers were introduced by extending the meaning of subtraction.

6. We now enlarge the meaning of powers by assuming that the principle

$$\frac{a^m}{a^n} = a^{m-n}$$

holds also when $m = n$.

We then have $$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But since $$\frac{a^m}{a^m} = 1,$$

it follows that $$a^0 = 1.$$

That is, *the zeroth power of any base, except* 0, *is equal to* 1.

E.g., $1^0 = 1$, $5^0 = 1$, $99^0 = 1$, $(a+b)^0 = 1$, etc.

7. Thus, by the assumption that the stated law holds when $m = n$, a definite value of the zeroth power of a number is obtained. Nevertheless, it will doubtless seem strange to the student that all numbers to the zeroth power have one and the same value, namely 1. But it should be distinctly noted that a^0 is by definition a symbol for $\frac{a^m}{a^m}$; *i.e.*, for the quotient of two like powers of the same base. Thus,

$$2^0 = \frac{2^3}{2^3} = \frac{2^5}{2^5} = \frac{2^m}{2^m} = 1.$$

Negative Integral Powers.

8. We now still further enlarge the meaning of powers by assuming that the principle

$$\frac{a^m}{a^n} = a^{m-n}$$

holds not only when $m > n$ and $m = n$, but also when $m < n$. In this case, $m - n$ is a negative number.

Since $m < n$, we may assume $n = m + k$.

Then $$\frac{a^m}{a^n} = \frac{a^m}{a^{m+k}} = a^{m-(m+k)} = a^{-k}.$$

But $$\frac{a^m}{a^{m+k}} = \frac{1}{a^{m+k-m}} = \frac{1}{a^k}.$$

Therefore $$a^{-k} = \frac{1}{a^k}.$$

That is, *a power with a negative exponent is equal to* 1 *divided by a power of the same base with a positive exponent of the same absolute value as the given exponent.*

E.g., $$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

Observe that the words *negative integral* refer only to the exponent, and not to the value of the power.

We are thus led to a quite definite and intelligible meaning of *negative powers* by extending still further the application of the stated law to the case in which $m < n$.

9. From the result of the preceding article we derive the following:

$$\left(\frac{a}{b}\right)^{-k} = \frac{1}{\left(\frac{a}{b}\right)^{k}} = \frac{1}{\frac{a^k}{b^k}} = \frac{b^k}{a^k} = \left(\frac{b}{a}\right)^{k}.$$

That is, *a negative integral power of any base is equal to a positive power of the reciprocal of the base, the exponents of the powers having the same absolute value; and vice versa.*

This reciprocal relation between positive and negative powers is useful in reductions which involve negative powers.

E.g., $$\left(\tfrac{2}{3}\right)^{-2} = \left(\tfrac{3}{2}\right)^{2} = \tfrac{9}{4}.$$

10. We also have $$\frac{1}{a^{-k}} = \frac{1}{\frac{1}{a^k}} = a^k.$$

This relation and the relation which defined a negative integral power may be stated thus:

Any power of a number may be transferred from the denominator to the numerator, or from the numerator to the denominator, of a fraction, if the sign of its exponent be reversed.

E.g., $$\frac{a^2}{a^{-3}} = a^2 \cdot a^3 = a^5; \quad \frac{(-a)^{-4}}{a} = \frac{1}{a(-a)^4} = \frac{1}{a^5}.$$

11. A negative integral power of zero is equal to an infinite.

For $$0^{-n} = \frac{1}{0^n} = \frac{1}{0} = \infty.$$

EXERCISES I.

Find the value of each of the following expressions:

1. 2^{-3}.
2. 3^{-2}.
3. $(\frac{2}{3})^{-1}$.
4. $(3\frac{3}{4})^{-3}$.
5. $(\frac{1}{3})^{-3}$.
6. $\dfrac{1}{.25^{-4}}$.
7. $\dfrac{1}{.2^{-6}}$.
8. $(2^{3})^{-6}$.

Change each of the following expressions into an equivalent expression in which all the exponents are positive:

9. x^3y^{-4}.
10. $2\,c^{-4}d$.
11. $3^{-1}a^2n^{-3}$.
12. $5\,x^{-2}y^{-3}$.
13. $\dfrac{2\,n^{-3}}{a^{-1}b^2}$.
14. $\dfrac{3\,b^2}{4\,a^{-5}c}$.
15. $\dfrac{5\,ad^{-2}}{7^{-1}b^{-3}c}$.
16. $\dfrac{3\,a^{-2}n^{-2}}{8\,b^{-4}}$.
17. $7 \times 3^{-2}ab^{-4}c^3d^{-5}$.
18. $9 \times 10^{-2}(\frac{3}{5})^{-3}ax^{-5}$.

In each of the following expressions transfer the factors from the denominator to the numerator:

19. $\frac{a}{b^2}$. **20.** $\frac{2x^2}{5y^{-3}}$. **21.** $\frac{3x^{-3}}{2^{-2}y}$. **22.** $\frac{5xy}{ab}$.

23. $\frac{3}{(a+b)}$. **24.** $\frac{4(x+y)^3}{(x-y)^2}$. **25.** $\frac{2a(x^2+1)}{3a^{-1}(x^2-1)^3}$.

26–32. In the examples 19–25 transfer the factors from the numerators to the denominators.

Fractional (Positive or Negative) Powers.

12. The meaning of a fractional power in which the exponent is the reciprocal of a positive integer will be determined first, then that of any fractional power. The word *fractional* refers to the exponent of the power and not to its value.

13. We will define, *i.e.*, fix the meaning of, the power $a^{\frac{1}{q}}$, in which q is a positive integer, by assuming that it must obey the first law of exponents, namely,

$$a^m \cdot a^n = a^{m+n}.$$

In other words, whatever meaning $a^{\frac{1}{q}}$ may have must be derived by an application of this law.

By this law, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$

But, since $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2$, by definition of positive integral power of *any* base, we have

$$(a^{\frac{1}{2}})^2 = a.$$

That is, $a^{\frac{1}{2}}$ is a number whose *square is a*, or $a^{\frac{1}{2}} = \sqrt{a}$.

In general,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots q \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\cdots q \text{ terms}} = a^{q \cdot \frac{1}{q}} = a.$$

But, since $a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots q$ factors $= (a^{\frac{1}{q}})^q$, by definition of positive integral power, we have $(a^{\frac{1}{q}})^q = a$.

That is, $a^{\frac{1}{q}}$ is a number whose *qth power is a*,

or $$a^{\frac{1}{q}} = \sqrt[q]{a}.$$

We are thus led, by the definition of the fractional power, $a^{\frac{1}{q}}$, to the operation that is inverse to that of raising a number to a positive integral power, *i.e.*, to the operation of finding a root.

Thus, $9^{\frac{1}{2}}$ and $\sqrt{9}$, $(-243)^{\frac{1}{5}}$ and $\sqrt[5]{-243}$, $a^{\frac{1}{q}}$ and $\sqrt[q]{a}$, are only different ways of representing the same numbers.

Notice that the index of the root is the *denominator* of the exponent of the fractional power, and the radicand is the *base*.

14. From the definition of a fractional power we have

$$(9^{\frac{1}{2}})^2 = (\sqrt{9})^2 = 9, \quad [(-25)^{\frac{1}{3}}]^3 = (\sqrt[3]{-25})^3 = -25.$$

In general, $$(a^{\frac{1}{q}})^q = (\sqrt[q]{a})^q = a. \tag{1}$$

Also, from Ch. XVI., § 1, Art. 10, $(a^q)^{\frac{1}{q}} = \sqrt[q]{a^q} = a$, if only principal roots be considered.

Therefore $$(a^{\frac{1}{q}})^q = (a^q)^{\frac{1}{q}},$$
for the principal root.

15. Meaning of $a^{\frac{p}{q}}$, wherein $\frac{p}{q}$ is a *positive or a negative* fraction. We may always assume q to be positive and p to have the sign of the fraction.

Whatever meaning $a^{\frac{p}{q}}$ may have must be derived by an application of the law

$$a^m \cdot a^n = a^{m+n}.$$

By this law, $$5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = 5^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 5^2.$$

But, since $5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = (5^{\frac{2}{3}})^3$, we have $(5^{\frac{2}{3}})^3 = 5^2$.

That is, $5^{\frac{2}{3}}$ is a number whose *cube is* 5^2; or $5^{\frac{2}{3}} = \sqrt[3]{5^2}$.

In general,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots, q \text{ terms}} = a^{q \cdot \frac{p}{q}}, = a^p.$$

But, since $a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots q$ factors $= (a^{\frac{p}{q}})^q$, we have $(a^{\frac{p}{q}})^q = a^p$.

That is, $a^{\frac{p}{q}}$ is a number whose *qth power is* a^p;

or $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Notice that a fractional power is a root of an integral power. The denominator of the fractional exponent is the index of the root, and the numerator is the exponent of the power.

E.g., $23^{\frac{4}{5}} = \sqrt[5]{23^4}$; $(-19)^{\frac{2}{3}} = \sqrt[3]{(-19)^2}$; $2^{-\frac{2}{3}} = \sqrt[3]{2^{-2}} = \sqrt[3]{\frac{1}{4}}$.

16. Since fractional powers simply afford another way of indicating roots, all the principles relating to roots which were proved in Chapters XVI. and XIX.–XX. hold for such powers.

EXERCISES II.

Write each of the following expressions as an equivalent expression with radical signs:

1. $a^{\frac{1}{2}}$. **2.** $b^{-\frac{1}{4}}$. **3.** $x^{\frac{2}{3}}$. **4.** $3\,y^{\frac{1}{2}}$.

5. $4\,x^{-\frac{3}{2}}y^{\frac{1}{2}}$. **6.** $2\,ab^{-\frac{5}{6}}c$. **7.** $2^{\frac{1}{2}}x^{\frac{3}{4}}y^{\frac{1}{6}}$. **8.** $2\,a^{\frac{m}{n}}b^{-\frac{p}{q}}$.

9. $\left(\frac{a}{b}\right)^{\frac{4}{5}}$. **10.** $\left(\frac{2\,x}{3\,y}\right)^{-\frac{5}{8}}$. **11.** $\frac{4\,m^{\frac{4}{5}}}{3\,n^{\frac{5}{6}}}$. **12.** $\frac{ab^{-\frac{m}{n}}}{xy^{-\frac{p}{q}}}$.

Find the value of each of the following expressions:

13. $4^{\frac{1}{2}}$. **14.** $169^{\frac{1}{2}}$. **15.** $16^{-\frac{1}{2}}$. **16.** $144^{-\frac{1}{2}}$.

17. $27^{\frac{1}{3}}$. **18.** $27^{-\frac{1}{3}}$. **19.** $16^{\frac{1}{4}}$. **20.** $81^{-\frac{1}{4}}$.

21. $49^{\frac{3}{2}}$. **22.** $512^{\frac{2}{3}}$. **23.** $216^{-\frac{5}{3}}$. **24.** $32^{-\frac{3}{5}}$.

25. $64^{\frac{2}{3}}$. **26.** $64^{\frac{3}{2}}$. **27.** $.09^{\frac{3}{2}}$. **28.** $(3\frac{3}{8})^{-\frac{2}{3}}$.

Write each of the following expressions as an equivalent expression with fractional exponents:

29. $\sqrt{a}$. **30.** $\sqrt{a^3}$. **31.** $\sqrt{(a^{-3}b^7)}$. **32.** $\sqrt{(2\,xy^{-5})}$.

33. $\sqrt[3]{a^2}$. **34.** $\sqrt[3]{(2\,x^{-1}y^2)}$. **35.** $\sqrt[4]{(5\,x^{-2}y^5)}$. **36.** $\sqrt[5]{(3\,a^{-7}b^6)}$.

37. $\sqrt[n]{(3\,a^2)}$. **38.** $\sqrt[q]{(3\,a^{-n})}$. **39.** $\sqrt[q]{[(a+b)^2(x-y)^{-p}]}$.

Simplify each of the following expressions:

40. $36^{\frac{1}{2}} + 8^{\frac{2}{3}} - 625^{.75}$. **41.** $.16^{.5} - (1\frac{217}{512})^{\frac{2}{3}} + 64^{-\frac{1}{2}}$.

42. $2\,a^{-\frac{3}{4}} - .4\,a^{\frac{1}{4}} + 2.5\,a^{.75} - 5\,a^{.25}$.

17. The consideration of **Irrational Powers** is reserved for a subsequent chapter.

18. Having thus determined definite meanings for zeroth, negative, and fractional powers, it remains to prove that they obey all the principles of positive integral powers.

Products of Powers.

(I.) $$a^m a^n = a^{m+n},$$
for all rational values of m and n.

Ex. 1. $x^5 x^{-7} = x^{5+(-7)} = x^{5-7} = x^{-2} = \frac{1}{x^2}.$

Ex. 2. $a^{\frac{1}{2}} b^{-\frac{3}{4}} \times a^{-3} b^4 = a^{\frac{1}{2}-3} b^{-\frac{3}{4}+4} = a^{-\frac{5}{2}} b^{\frac{13}{4}} = \frac{b^{\frac{13}{4}}}{a^{\frac{5}{2}}}.$

(i.) m positive and n negative, and the absolute value of m less than the absolute value of n.

Let $n = -n_1$, so that n_1 is positive. Then

$$a^m a^n = a^m a^{-n_1} = \frac{a^m}{a^{n_1}} = \frac{1}{a^{n_1 - m}} = \frac{1}{a^{-[m+(-n_1)]}} = a^{m+(-n_1)} = a^{m+n}.$$

In a similar way the principle can be proved for other cases in which the exponents are 0 or negative.

That the principle holds when the exponents, either or both, are fractions, follows from the definition of a fractional power.

EXERCISES III.

Simplify each of the following expressions:

1. $x^3 x^0$. **2.** $x^{-3} x^3$. **3.** $a^{-5} a^6$. **4.** $m^{-3} m^{-5}$.

5. $a^3 a^{\frac{1}{2}}$. **6.** $a^{\frac{2}{3}} a^{\frac{3}{4}}$. **7.** $b^{-\frac{5}{6}} b^{\frac{2}{3}}$. **8.** $c^{-\frac{1}{3}} c^{-\frac{3}{8}}$.

9. $5\,a^{-3} \times 3\,a^5$. **10.** $-\frac{5}{7}\,b^{-2} \times 1\frac{2}{5}\,b^{-3}$. **11.** $3\frac{3}{4}\,x^3 \times 2\frac{2}{3}\,x^{-\frac{1}{2}}$.

12. $a^3 b^{-2} \times a^{\frac{1}{3}} b^{\frac{2}{3}}$. **13.** $3\,x^{-\frac{3}{4}} y^{\frac{2}{3}} \times 2\,x^{-5} y^{-\frac{1}{2}}$. **14.** $2\,a^{-\frac{1}{3}} b^{-2} c^3 \times 5\,a^6 c^{-\frac{7}{3}}$.

15. $a^m b^{-n} \times a^{-p} b^q$. **16.** $xy^p \times x^{\frac{m}{n}} y^{-\frac{p}{q}}$. **17.** $a^{\frac{1}{m-n}} b^{\frac{p}{q}} \times a^{\frac{1}{m+n}} b^{-\frac{p}{q}}$.

18. $\frac{12\,a^{-3}}{n^{-2}} \times \frac{a^2}{9\,n^3}$. **19.** $\frac{7\,c^{-3}}{3\,a^3} \div \frac{35\,a^{-4}}{6\,c^2}$. **20.** $\frac{a^{-m} b^{-n}}{\frac{1}{2}\,c} \div \frac{c^{-1}}{a^{-2n} b^{-2n}}$.

21. $(a^{\frac{1}{3}} + x^{-2})(a^{\frac{1}{3}} - x^{-2})$. **22.** $(a^{\frac{1}{2}} + a^{-\frac{1}{2}})(a^{\frac{1}{2}} - a^{-\frac{1}{2}})$.

23. $(a^{\frac{2}{3}} - a^{\frac{1}{3}} b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})$. **24.** $(x^2 y^{-\frac{2}{3}} + xy^{-\frac{1}{3}} + 1)(xy^{-\frac{1}{3}} - 1)$.

25. $(a^{\frac{1}{2}} + a^{\frac{1}{4}} b^{\frac{1}{4}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - a^{\frac{1}{4}} b^{\frac{1}{4}} + b^{\frac{1}{2}})$. **26.** $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}} b)(ab^{-\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

27. $(a^{-7} + a^{-5} - a^{-3})(a^7 + a^5 + a^3)$.

28. $(x^3 - x^{-3} - 2\,x^{-6} + 5)(10\,x^{-7} + x^{-1} - 5\,x^{-4})$.

29. $(\frac{1}{2}\,a^{-6} x^{-n} + \frac{3}{4}\,x^{-3n} - a^{-3} x^{-2n})(\frac{2}{3}\,a^2 x^p - 4\,a^{-4} x^{p+2n} + a^{-1} x^{p+n})$.

30. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}} y - y^{\frac{3}{2}})(x + x^{\frac{1}{2}} y^{\frac{1}{2}} + y)$.

31. $(a^{\frac{2}{3}} + a^{-\frac{2}{3}} - a^{\frac{1}{3}} - a^{-\frac{1}{3}})(a^{\frac{1}{3}} + a^{-\frac{1}{3}} + 1)$.

32. $(x^{\frac{2}{3}} + 2\,x^{.5} + 3\,x^{\frac{1}{3}} + 2\,x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - 2\,x^{\frac{1}{6}} + 1)$.

33. $(a^{-1.5} + b^{-1.6} - a^{-.75}b^{-.8})(a^{-.75} + b^{-.8})$.

34. $(1\frac{4}{5}\,a^{.5}x^{1\frac{1}{5}} + 2\,a^{\frac{3}{2}} + \frac{2}{5}\,x^{1.8} + 6\,ax^{\frac{3}{5}})(a^{\frac{1}{2}} - 3\,x^{\frac{3}{5}} + \frac{5}{2}\,x^{1\frac{4}{5}}a^{-1})$.

Quotients of Powers.

(II.)
$$\frac{a^m}{a^n} = a^{m-n},$$

for all rational values of m and n.

Ex. 1.
$$\frac{x^2}{x^{-3}} = x^{2-(-3)} = x^{2+3} = x^5.$$

Ex. 2.
$$\frac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{a^{\frac{1}{4}}b^{-\frac{3}{2}}} = a^{-\frac{1}{2}-\frac{1}{4}}b^{\frac{2}{3}+\frac{3}{2}} = a^{-\frac{3}{4}}b^{\frac{13}{6}} = \frac{b^{\frac{13}{6}}}{a^{\frac{3}{4}}}.$$

We have
$$\frac{a^m}{a^n} = a^m a^{-n} = a^{m+(-n)} = a^{m-n}.$$

EXERCISES IV.

Simplify each of the following expressions:

1. $\frac{a}{a^{-1}}$. **2.** $\frac{x^0}{x^{-2}}$. **3.** $\frac{5^{-2}}{5^{-3}}$. **4.** $\frac{a^2}{a^{\frac{1}{2}}}$. **5.** $\frac{x^{-2}}{x^{-5}}$. **6.** $\frac{x^{\frac{2}{3}}}{x^{-\frac{1}{6}}}$.

7. $\frac{a^{-\frac{3}{4}}}{a^{-\frac{1}{4}}}$. **8.** $\frac{a^{\frac{2}{3}}}{a^{-2}}$. **9.** $\frac{x^n}{x^{-n}}$. **10.** $\frac{x^{m-n}}{x^{-n}}$. **11.** $\frac{x^{-1}}{x^{n-1}}$. **12.** $\frac{x^{5-n}}{x^{-5}}$.

13. $(1\frac{1}{2}\,b^{-3}) \div (3\,b^2)$. **14.** $1 \div (\frac{1}{2}\,ab^{-1})$. **15.** $(3\frac{1}{2}\,a^n b^{-4}) \div (\frac{7}{8}\,a^n b^{-3})$.

16. $12\,a^{-1}b^{-1}x^{-\frac{3}{4}} \div \frac{4\,a^2b^{-\frac{3}{5}}}{x^{-\frac{2}{3}}}$. **17.** $\frac{2\,x^2y^{-\frac{1}{2}}}{3\,a^2b^{-4}} \times \frac{6\,a^{-5}b^{\frac{1}{2}}}{7\,x^3y^{\frac{3}{4}}}$.

18. $(a^{\frac{1}{2}} - b^{\frac{1}{2}}) \div (a^{\frac{1}{4}} + b^{\frac{1}{4}})$. **19.** $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$.

20. $(x^{-1} + y^{-1}) \div (x^{-\frac{1}{3}} + y^{-\frac{1}{3}})$. **21.** $(x^{-4} - y^{-4}) \div (x^{-1} + y^{-1})$.

22. $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y) \div (x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}})$. **23.** $(a^{\frac{3n}{2}} - a^{-\frac{3n}{2}}) \div (a^{\frac{n}{2}} - a^{-\frac{n}{2}})$.

24. $[(a - 1)^{-2} - 1] \div [(a - 1)^{-1} - 1]$.

25. $(3\,a^{-10} + a^6 - 4\,a^{-6}) \div (2\,a^{-2} + a^2 + 3\,a^{-6})$.

26. $(2\,x^{-3} - 3\,x^{-2} - 2\,x^{-1} + 2 - x) \div (x^{-1} + 1)$.

27. $(x^{-1} - 3\,x^{-\frac{1}{2}} + 3 - 3\,x^{\frac{1}{2}} + 2\,x) \div (x^{-\frac{3}{2}} - 2\,x^{-1} + x^{-\frac{1}{2}} - 2)$.

28. $(2a^7 - 3a^8 - 23a^{-1} + 15a^{-5} + 9a^{-9}) \div (a^4 + 2 - 3a^{-4})$.

29. $(6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} - 2x^{-1} - 13) \div (3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - 5)$.

30. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

31. $(x^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + a^{\frac{4}{3}}) \div (x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}})$.

32. $(a^{\frac{5}{2}} - a^2b^{\frac{1}{2}} - a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}) \div (a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}})$.

33. $(6x^{\frac{5}{4}} - 7x - 19x^{\frac{3}{4}} + 2x^{\frac{1}{2}} + 8x^{\frac{1}{4}}) \div (2x^{\frac{3}{4}} - 3x^{\frac{1}{2}} - 4x^{\frac{1}{4}})$.

Powers of Powers.

$$\text{(III.)} \qquad (a^m)^n = a^{mn},$$

for all rational values of m and n.

Ex. 1. $(x^2)^{-3} = x^{2(-3)} = x^{-6} = \frac{1}{x^6}$.

Ex. 2. $(1024^{\frac{1}{2}})^{-\frac{3}{5}} = 1024^{-\frac{3}{10}} = \frac{1}{(\sqrt[10]{1024})^3} = \frac{1}{8}$.

Ex. 3. $(1 + x^{-\frac{1}{2}})^2 = 1 + 2x^{-\frac{1}{2}} + (x^{-\frac{1}{2}})^2 = 1 + \frac{2}{\sqrt{x}} + \frac{1}{x}$.

(i.) m and n both negative integers.

Let $m = -m_1$ and $n = -n_1$, so that m_1 and n_1 are positive.

We have

$$(a^m)^n = (a^{-m_1})^{-n_1} = \left(\frac{1}{a^{m_1}}\right)^{-n_1} = (a^{m_1})^{n_1} = a^{m_1n_1} = a^{(-m_1)(-n_1)} = a^{mn}.$$

In a similar manner the principle can be proved for other cases in which the exponents are 0 or negative integers.

(ii.) m a fraction, and n a positive or a negative integer, or 0.

Let $m = \frac{p}{q}$, wherein q is a positive integer and p is a positive or a negative integer.

We then have $(a^m)^n = (a^{\frac{p}{q}})^n = [(a^{\frac{1}{q}})^p]^n = (a^{\frac{1}{q}})^{pn} = a^{\frac{pn}{q}} = a^{\frac{p}{q}n} = a^{mn}$.

In a similar manner the principle can be proved when m is an integer and n is a fraction.

(iii.) m and n both fractions.

Let $m = \frac{p}{q}$ and $n = \frac{r}{s}$, wherein q and s are positive integers, and p and r are positive or negative integers.

If $(a^{\frac{p}{q}})^{\frac{r}{s}}$ be raised to the qsth, $= sq$th power, we have

$$[(a^{\frac{p}{q}})^{\frac{r}{s}}]^{qs} = \{[(a^{\frac{p}{q}})^{\frac{r}{s}}]^{s}\}^{q} = [(a^{\frac{p}{q}})^{r}]^{q} = [(a^{\frac{p}{q}})^{q}]^{r} = (a^{p})^{r} = a^{pr}.$$

Consequently $(a^{\frac{p}{q}})^{\frac{r}{s}}$ is the qs root of a^{pr}; or, by definition of a fractional power,

$$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}} = a^{\frac{p}{q}\cdot\frac{r}{s}}.$$

EXERCISES V.

Simplify each of the following expressions:

1. $(x^2)^{-2}$. 2. $(a^3)^{\frac{1}{2}}$. 3. $[(-x)^{\frac{1}{3}}]^2$. 4. $(x^{-3})^4$.

5. $(x^{-\frac{2}{5}})^{15}$. 6. $(a^{-3})^{\frac{1}{6}}$. 7. $(b^3)^{-\frac{4}{9}}$. 8. $(x^{-2})^{-5}$.

9. $(x^{-\frac{1}{3}})^{-\frac{1}{2}}$. 10. $(a^n)^{-2}$. 11. $(a^{-m})^{-3}$. 12. $(a^{-\frac{p}{q}})^{-\frac{m}{n}}$.

13. $(\sqrt[3]{a^{-2}})^4$. 14. $(\sqrt{a})^{-\frac{3}{2}}$. 15. $(\sqrt[5]{x^{\frac{4}{3}}})^{-\frac{3}{2}}$. 16. $(\sqrt[q]{a^{-m}})^{-3}$.

17. $(a^{\frac{1}{2}} - a^{-\frac{1}{2}})^2$. 18. $(1 - x^{-\frac{1}{4}})^2$. 19. $(a^{\frac{1}{3}} + a^{-\frac{1}{3}})^3$.

Powers of Products.

(IV.) $(ab)^m = a^m b^m$, for all rational values of m.

Ex. 1. $(2x)^{-3} = 2^{-3}x^{-3} = \dfrac{1}{8x^3}$.

Ex. 2. $(3x^{-\frac{1}{2}}y^2)^{-4} = 3^{-4}x^2y^{-8} = \dfrac{x^2}{81y^8}$.

(i.) m a negative integer. Let $m = -m_1$, so that m_1 is positive.

Then $(ab)^m = (ab)^{-m_1} = \dfrac{1}{(ab)^{m_1}} = \dfrac{1}{a^{m_1}b^{m_1}} = a^{-m_1}b^{-m_1} = a^m b^m$.

(ii.) m a fraction. Let $m = \dfrac{p}{q}$, where p is a positive or negative integer, and q is a positive integer.

If $(ab)^{\frac{p}{q}}$ be raised to the qth power, we have

$$[(ab)^{\frac{p}{q}}]^q = (ab)^p, \text{ since } q \text{ is an integer,}$$
$$= a^p b^p, \text{ by (i.).}$$

But $\quad (a^{\frac{p}{q}}b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q(b^{\frac{p}{q}})^q = a^p b^p.$

Therefore $\quad [(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}}b^{\frac{p}{q}})^q$; whence $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}}$.

Powers of Quotients.

(V.) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, for all rational values of m.

Ex. 1. $\left(\frac{a^{\frac{1}{2}}}{b^3}\right)^{-3} = \frac{a^{-\frac{3}{2}}}{b^{-9}} = \frac{b^9}{a^{\frac{3}{2}}}$. **Ex. 2.** $\left(\frac{4^{-3}}{x^2y^{-1}}\right)^{-\frac{1}{2}} = \frac{4^{\frac{3}{2}}}{x^{-1}y^{\frac{1}{2}}} = \frac{8x}{y^{\frac{1}{2}}}$.

We have $\left(\frac{a}{b}\right)^m = (ab^{-1})^m = a^m b^{-m} = \frac{a^m}{b^m}$.

EXERCISES VI.

Simplify each of the following expressions :

1. $(a^{\frac{1}{2}}x^{-1})^{-2}$.
2. $(\frac{1}{4}a)^{-\frac{1}{2}}$.
3. $(8a^{-6})^{\frac{1}{3}}$.
4. $(a^{-1}b^{-3})^{-4}$.
5. $(2a^{\frac{3}{2}}x)^{\frac{5}{6}}$.
6. $(x^{\frac{1}{3}}a^{-\frac{1}{2}})^{-12}$.
7. $(\frac{1}{32}x^{-10})^{-\frac{1}{5}}$.
8. $(a^{-2}b^{\frac{1}{2}}c^{-\frac{3}{4}})^{-6}$.
9. $(2a^{-1}\sqrt{x})^{-2}$.
10. $(5x^{-3}\sqrt[3]{a^2})^{-6}$.
11. $(3\sqrt{x}\sqrt[3]{x^{-1}})^{-6}$.
12. $[3\sqrt[n]{(a^{-m}b^p)}]^{-mp}$.
13. $\left(\frac{x^{\frac{2}{3}}}{y^{-\frac{1}{4}}}\right)^{-6}$.
14. $\left(\frac{-2^3a^{-3}}{4b^3}\right)^{-2}$.
15. $\left(\frac{4x^{-\frac{1}{2}}}{y^5}\right)^{-\frac{1}{5}}$.
16. $\left(\frac{8a^2}{27a^{-3}y^{\frac{1}{3}}}\right)^{-\frac{1}{3}}$.
17. $\left(\frac{2x^{\frac{4}{5}}}{3a^{-2}b^2}\right)^{-5}$.
18. $\left(\frac{16a^{-8}}{81x^2y^{-1}}\right)^{-\frac{1}{4}}$.
19. $\left(\frac{5a^{-\frac{1}{6}}b^{\frac{1}{3}}}{6x^{-2}}\right)^3$.
20. $\left(\frac{2x^{-\frac{1}{10}}y^2}{3a^{-4}b^{-\frac{3}{5}}}\right)^5$.
21. $\left(\frac{\sqrt{a}}{\sqrt[3]{x^2}}\right)^{-6}$.
22. $\left(\frac{2\sqrt[3]{a^{-2}}}{3\sqrt{b^{-3}}}\right)^6$.
23. $\left(\frac{3\sqrt[4]{x^8}}{5\sqrt{a^{-8}}}\right)^2$.
24. $\left(\frac{7\sqrt[4]{x^{-\frac{2}{3}}}}{8\sqrt[3]{y^{-2}}}\right)^{-3}$.
25. $\left[\left(\frac{2a^{-\frac{1}{3}}}{x^{-1}y^{-\frac{1}{6}}}\right)^2\right]^{-3}$.
26. $\left[\left(\frac{4a^{-\frac{1}{4}}x^2}{9b^{-3}y^{\frac{1}{2}}}\right)^{-\frac{1}{3}}\right]^2$.
27. $\left[\left(\frac{\sqrt[4]{a^{-8}}}{2x^{-\frac{1}{2}}y^{\frac{5}{6}}}\right)^{-\frac{2}{3}}\right]^{-6}$.
28. $(a^{\frac{1}{2}}b^{-2}-1)^2$.
29. $(a^{-\frac{3}{2}}b^{-1}+ab^{\frac{3}{2}})^2$.
30. $(a^{-3}b^{\frac{1}{3}}-c^{-\frac{2}{3}})^3$.

EXERCISES VII.

MISCELLANEOUS EXAMPLES.

Simplify each of the following expressions:

1. $(-\sqrt[3]{x^{\frac{2}{5}}})^5 + (-2\sqrt[3]{x^{\frac{1}{2}}})^4 - x^{-1}\left(\frac{-3x^{\frac{2}{3}}}{\sqrt[3]{x^{-.5}}}\right)^2$.
2. $4.5(\sqrt[3]{a^{10}})^{\frac{3}{20}} + \left(\sqrt[3]{\frac{4}{a}}\right)^{-1\frac{1}{2}} - \left(\frac{3.375}{\sqrt[4]{a^{-8}}}\right)^{\frac{2}{3}}$.

3. $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}}+\dfrac{1}{(1-x^2)^{\frac{1}{2}}}.$

4. $\dfrac{a^{1\frac{1}{2}}}{a^{1\frac{1}{2}}+x^{\frac{1}{4}}}+\dfrac{x^{\frac{1}{4}}}{a^{1\frac{1}{2}}-x^{\frac{1}{4}}}.$

5. $\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}.$

6. $\dfrac{a-x}{a^{\frac{1}{3}}-x^{\frac{1}{3}}}-\dfrac{a+x}{a^{\frac{1}{3}}+x^{\frac{1}{3}}}.$

7. $\dfrac{a^{\frac{1}{2}}x^{\frac{1}{4}}+a^{\frac{1}{4}}x^{\frac{1}{2}}}{a^{\frac{1}{2}}+x^{\frac{1}{2}}}\cdot\dfrac{a-x}{a^{\frac{1}{4}}+x^{\frac{1}{4}}}.$

8. $\dfrac{x^{\frac{1}{2}}+1}{x+x^{\frac{1}{2}}+1}\div\dfrac{1}{x^{1.5}-1}.$

9. $\dfrac{1}{a^{\frac{1}{4}}+a^{\frac{1}{8}}+1}+\dfrac{1}{a^{\frac{1}{4}}-a^{\frac{1}{8}}+1}-\dfrac{2a^{\frac{1}{4}}}{a^{\frac{1}{2}}-a^{\frac{1}{4}}+1}.$

Find the square root of each of the following expressions:

10. $x^{\frac{1}{2}}+x^{-\frac{1}{2}}+2.$

11. $a^{-4}x+2a^{-\frac{3}{2}}x^{-\frac{3}{2}}+ax^{-4}.$

12. $4x^{-4}-12x^{-3}+13x^{-2}-6x^{-1}+1.$

13. $9x^2+10x^{-2}-4x^{-4}+x^{-6}-12.$

14. $a^2-\frac{3}{2}a^{\frac{3}{2}}-\frac{3}{2}a^{\frac{1}{2}}+\frac{41}{16}a+1.$

15. $\frac{9}{4}x^3-5x^{\frac{5}{2}}y^{\frac{1}{2}}+\frac{179}{45}x^2y-\frac{4}{3}x^{\frac{3}{2}}y^{\frac{3}{2}}+\frac{4}{25}xy^2.$

Find the cube root of each of the following expressions:

16. $x^{-6}-6x^{-5}+12x^{-4}-8x^{-3}.$

17. $8x-36x^{\frac{7}{6}}-27x^{\frac{3}{2}}+54x^{\frac{4}{3}}.$

18. $8x^{-3}+12x^{-2}-30x^{-1}-35+45x+27x^2-27x^3.$

19. $x^{\frac{3}{2}}-3x^{\frac{4}{3}}+3x^{\frac{7}{6}}+2x+3x^{\frac{5}{4}}-3x^{\frac{5}{6}}-6x^{\frac{13}{12}}+3x^{\frac{11}{12}}+x^{\frac{3}{4}}.$

20. $8x^3y^{-\frac{3}{2}}+13x^{\frac{3}{2}}+y^{\frac{3}{2}}+12x^{\frac{5}{2}}y^{-1}+18x^2y^{-\frac{1}{2}}+9xy^{\frac{1}{2}}+3x^{\frac{1}{2}}y.$

Solve each of the following equations:

21. $2x^{-2}-x^{-1}-1=0.$

22. $x^{-2}+5x^{-1}-\frac{11}{4}=0.$

23. $3x^{-6}-2x^{-3}-1=0.$

24. $5x-2x^{\frac{1}{2}}-16=0.$

25. $3x^{\frac{1}{2}}-2x^{\frac{1}{4}}-4=0.$

26. $5x^{-\frac{1}{2}}-2x^{-\frac{1}{4}}-3=0.$

27. $x^{\frac{2}{3}}+3x^{\frac{1}{3}}-10=0.$

28. $7x^{-\frac{4}{3}}-3x^{-\frac{2}{3}}-4=0.$

29. $5x^{\frac{2}{5}}-3x^{\frac{1}{5}}-14=0.$

30. $2x^{-\frac{4}{5}}+5x^{-\frac{2}{5}}-7=0.$

CHAPTER XXVII.

PROGRESSIONS.

§ 1.

1. A **Series** is a succession of numbers, each formed according to some definite law. The single numbers are called the **Terms** of the series.

The law may specify that each term shall be formed from the immediately preceding term in a prescribed way.

E.g., in the series

$$1 + 3 + 5 + 7 + 9 + \cdots \qquad (1)$$

each term after the first is formed by adding 2 to the preceding term.

In the series $1 + 2 + 4 + 8 + \cdots \qquad (2)$

each term after the first is formed by multiplying the preceding term by 2.

Or the law may state a definite relation between each term and the number of its place in the series.

E.g., in the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad (3)$

each term is the reciprocal of its number.

2. The number of terms in a series may be either *limited* or *unlimited.*

A **Finite** series is one of a *limited* number of terms.

An **Infinite** series is one of an *unlimited* number of terms.

In this chapter a few simple and yet very important series will be discussed.

§ 2. ARITHMETICAL PROGRESSION.

1. An **Arithmetical Series,** or as it is more commonly called an **Arithmetical Progression** (A. P.), is a series in which each term, after the first, is formed by adding a constant number to the preceding term. See § 1, Art. 1, (1).

Evidently this definition is equivalent to the statement, that the difference between any two consecutive terms is constant.

E.g., in the series

$$1 + 3 + 5 + 7 + \cdots$$

we have $$3 - 1 = 5 - 3 = 7 - 5 = \cdots$$

For this reason the constant number of the first definition is called the **Common Difference** of the series.

2. Let a_1 stand for the first term of the series,

a_n for the nth (*any*) term of the series,

d for the common difference,

and S_n for the sum of n terms of the series.

The five numbers a_1, a_n, d, n, S_n are called the **Elements** of the progression.

3. The common difference may be either positive or negative.

If d be *positive*, each term is greater than the preceding, and the series is called a *rising*, or an *increasing* progression.

E.g., $1 + 2 + 3 + 4 + \cdots$, wherein $d = 1$.

If d be negative, each term is less than the preceding, and the series is called a *falling*, or a *decreasing* progression.

E.g., $1 - 1 - 3 - 5 - \cdots$, wherein $d = -2$.

4. *In an arithmetical progression any term is equal to the first term plus the product of the common difference and a number one less than the number of the required term,* i.e.,

$$a_n = a_1 + (n - 1)d. \qquad \text{(I.)}$$

By the definition of an arithmetical progression

$$a_1 = a_1,\ \ a_2 = a_1 + d,\ \ a_3 = a_2 + d = a_1 + 2d, \text{ etc.}$$

The law expressed by the formulæ for these first three terms is evidently general, and since the coefficient of d in each is one less than the number of the corresponding term, we have

$$a_n = a_1 + (n - 1)d.$$

Ex. 1. Find the 15th term of the progression

$$1+3+5+7+\cdots$$

we have $a_1=1,\ d=2,\ n=15;$

therefore $a_{15}=1+(15-1)2=1+28=29.$

This formula may be used not only to find a_n, when a_1, d, and n are given, but also to find any one of the four numbers involved when the other three are given.

Ex. 2. If $a_5=3\ (n=5)$, and $a_1=1$, we have $3=1+4d$; whence $d=\frac{1}{2}$.

5. *In any arithmetical progression, the sum of n terms is equal to one-half the product of the number of terms and the sum of the first and the nth term,* i.e.,

$$S_n=\frac{n}{2}(a_1+a_n). \qquad \text{(II.)}$$

Since the successive terms in an arithmetical progression, from the first to the nth inclusive, may be obtained either by repeated additions of the common difference beginning with the first term, or by repeated subtractions of the common difference beginning with the nth term, we may express the sum of n terms in two equivalent ways:

$$S_n=a_1+(a_1+d)+(a_1+2d)+\cdots+(a_1+\overline{n-2}\cdot d)+(a_1+\overline{n-1}\cdot d),$$

$$S_n=a_n+(a_n-d)+(a_n-2d)+\cdots+(a_n-\overline{n-2}\cdot d)+(a_n-\overline{n-1}\cdot d).$$

Whence, by addition,

$$2S_n=(a_1+a_n)+(a_1+a_n)+\cdots+(a_1+a_n)+(a_1+a_n),$$

wherein there are n binomials a_1+a_n.

Therefore, $2S_n=n(a_1+a_n)$, or $S_n=\frac{n}{2}(a_1+a_n)$.

6. If the value of a_n, given in (I.), be substituted for a_1 in (II.), we obtain

$$S_n=\frac{n}{2}[2a_1+(n-1)d]. \qquad \text{(III.)}$$

Formula (II.) is used when a_1, a_n, and n are given; and (III.) when a_1, d, and n are given.

7. Ex. 1. If $a_1 = 1$, $a_5 = 3$, then $S_5 = \frac{5}{2}(1+3) = 10$.

Ex. 2. If $a_1 = -4$, $d = 2$, $n = 12$,

then $$S_{12} = \tfrac{12}{2}[2(-4) + 11 \times 2] = 84.$$

Either (II.) or (III.) can be used to determine any one of the five elements a_1, a_n, d, n, S_n, when the three others involved in the formula are known.

Ex. 3. Given $a_1 = -3$, $d = 2$, $S_n = 12$, to find n.

From (III.), $12 = \frac{n}{2}[-6 + 2(n-1)]$,

or $$n^2 - 4n = 12; \text{ whence } n = 6 \text{ and } -2.$$

The result 6 gives the series $-3 - 1 + 1 + 3 + 5 + 7$, $= 12$.

Since the number of terms must be positive, the negative result, -2, is not admissible. But its meaning may be assumed to be that two terms, beginning with the last and counting toward the first, are to be taken.

8. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five numbers a_1, a_n, d, S_n, n, when the three others are given.

Ex. 1. Given $d = \frac{1}{2}$, $n = 9$, $a_9 = 5$, to find a_1 and S_9.

From (I.), $\quad 5 = a_1 + 8 \cdot \frac{1}{2}$, whence $a_1 = 1$.

From (II.), using the value of a_1 just found,

$$S_9 = \tfrac{9}{2}(1+5) = 27.$$

Ex. 2. Given $a_1 = 3$, $n = 13$, $S_{13} = 13$, to find d and a_{13}.

From (II.), $13 = \frac{13}{2}(3 + a_{13})$, whence $a_{13} = -1$.

From (I.), $-1 = 3 + 12d$, whence $d = -\frac{1}{3}$.

Ex. 3. Given $d = -2$, $a_n = -16$, $S_n = -60$, to find a_1 and n.

From (I.), $$-16 = a_1 - 2(n-1), \tag{1}$$

and from (II.), $$-60 = \frac{n}{2}(a_1 - 16). \tag{2}$$

Solving (1) and (2), we obtain $n = 12$, $a_1 = 6$; and $n = 5$, $a_1 = -8$.

The two series are:

$$6+4+2+0-2-4-6-8-10-12-14-16,$$

and

$$-8-10-12-14-16,$$

both of which have $d=-2,\ a_n=-16,\ S_n=-60.$

Notice that in this example the sum of the terms which are not common to the two series is 0.

Ex. 4. Given $a_1=1,\ d=2,\ a_n=18$, to find n.

From (I.), $18=1+2(n-1)$, whence $n=\frac{19}{2}$.

This result is inadmissible.

A glance at the data will reveal the meaning of this result. Since the first term is odd and the common difference even, the last term cannot be even.

This example also illustrates the fact that, although one of the formulæ determines one of the elements when the other three elements are given, yet these three elements cannot be arbitrarily assumed.

9. In many examples the elements necessary for determining the required element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. 1. Given $a_6=17,\ a_{11}=32$, to find a_1 and d.

From (I.), $17=a_1+5\,d$, and $32=a_1+10\,d$.

Solving these equations, $a_1=2,\ d=3$.

Or, we could have regarded 17 as the first term and 32 as the last term of a progression of 6 terms. Then, by (I.), $32=17+5\,d$, whence $d=3$.

By (I.) again, $17=a_1+5\times 3$; whence $a_1=2$, as above.

Ex. 2. Given $S_8=80,\ S_{12}=168$; find a_1 and d.

From (III.), $80=\frac{8}{2}(2\,a_1+7\,d)$, and $168=\frac{12}{2}(2\,a_1+11\,d)$.

From these equations, $a_1=3,\ d=2$.

EXERCISES I.

Find the last term, and the sum of the terms, of each of the following arithmetical progressions:

1. $2 + 6 + \cdots$ to 10 terms.

2. $3 + 1 - \cdots$ to 13 terms.

3. $-5 - 2 + \cdots$ to 21 terms.

4. $13 + 11 + \cdots$ to 25 terms.

5. $3 + 1\frac{1}{2} + \cdots$ to 40 terms.

6. $4 + 1\frac{3}{4} - \cdots$ to 31 terms.

7. $9 + 11 + \cdots$ to n terms.

8. $1 + \frac{5}{6} + \cdots$ to n terms.

9. $n + 2n + \cdots$ to 16 terms, to m terms.

10. $a + (a + b) + \cdots$ to 20 terms, to n terms.

11. $x + (3x - 2y) + \cdots$ to 15 terms, to n terms.

12. $(m + 2) + (4m + 5) + \cdots$ to 40 terms, to n terms.

13. $\frac{a-1}{a} + \frac{a-3}{a} + \cdots$ to 30 terms, to n terms.

14. $\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \cdots$ to 32 terms, to n terms.

15. $n - 1 + \frac{n^2+1}{n-1} + \cdots$ to 11 terms, to x terms.

16. Find the sum of the series

$$(e^{x+1} - e^x + 1) + (e^{x+1} + e^x - 1) + \cdots + (e^{x+1} + 19e^x - 19).$$

In each of the following arithmetical progressions find the values of the two elements not given:

17. $a_1 = 4,\ d = 5,\ n = 10.$

18. $a_1 = 1.2,\ d = -3,\ n = 16.$

19. $a_n = -4,\ d = .3,\ n = 10.$

20. $a_n = 16,\ d = 2,\ n = 9.$

21. $a_1 = -5,\ n = 72,\ a_n = 37\frac{3}{5}.$

22. $a_1 = 2\frac{3}{5},\ n = 5,\ a_n = -1.9.$

23. $d = 6,\ n = 10,\ S_n = 340.$

24. $d = -4.8,\ n = 3,\ S_n = 28.5.$

25. $a_1 = 3,\ n = 15,\ S_n = 90.$

26. $a_n = 13,\ n = 8,\ S_n = 100.$

27. $a_1 = 4\frac{1}{3},\ n = 10,\ S_n = 73\frac{1}{3}.$

28. $a_n = 2\frac{1}{6},\ n = 12,\ S_n = -7.$

29. $a_1 = 9,\ d = -1,\ a_n = 6.$

30. $a_1 = 7,\ d = 5,\ a_n = 227.$

31. $a_1 = -7.5,\ a_n = 10.5,\ S_n = 15.$

32. $a_1 = 22\frac{1}{3},\ a_n = -19\frac{2}{3},\ S_n = 20.$

33. $a_1 = 2,\ d = 5,\ S_n = 245.$

34. $a_1 = -\frac{1}{4},\ d = \frac{2}{3},\ S_n = 282\frac{1}{2}.$

35. $a_n = 56,\ d = 5,\ S_n = 324.$

36. $a_n = 4\frac{2}{3},\ d = -\frac{1}{9},\ S_n = 69\frac{1}{3}.$

Arithmetical Means.

10. The **Arithmetical Mean** between two numbers is a third number, in value between the two, which forms with them an arithmetical progression.

E.g., 2 is an arithmetical mean between 1 and 3.

Let A stand for the arithmetical mean between a and b; then by the definition of an arithmetical progression,

$$A - a = b - A,$$

whence
$$A = \frac{a + b}{2}.$$

That is, the arithmetical mean between two numbers is half their sum.

11. Arithmetical Means between two numbers are numbers, in value between the two, which form with them an arithmetical progression.

E.g., 2, 3, and 4 are three arithmetical means between 1 and 5.

Ex. Insert four arithmetical means between -2 and 9.

We have $n = 6,\ a_1 = -2,\ a_6 = 9.$

From (I.), $9 = -2 + 5d$, whence $d = \frac{11}{5}$.

The required means are $\frac{1}{5}, \frac{12}{5}, \frac{23}{5}, \frac{34}{5}$.

12. If n arithmetical means be inserted between a and b, we have an arithmetical progression of $n + 2$ terms, the first term being a and the last b.

Therefore, from (I.), $b = a + (\overline{n + 2 - 1})d$,

whence
$$d = \frac{b - a}{n + 1}.$$

The resulting series is therefore

$$a,\ a + \frac{b - a}{n + 1},\ a + 2\frac{b - a}{n + 1},\ a + 3\frac{b - a}{n + 1},\ \cdots,\ b;$$

or
$$a,\ \frac{na + b}{n + 1},\ \frac{(n - 1)a + 2b}{n + 1},\ \frac{(n - 2)a + 3b}{n + 1},\ \cdots,\ b.$$

EXERCISES II.

Insert an arithmetical mean between

1. 45 and 31.
2. $17\frac{1}{2}$ and $14\frac{1}{2}$.
3. $2a$ and $-2b$.
4. $\frac{a - b}{a + b}$ and $\frac{a + b}{a - b}$.
5. $\frac{x + 1}{x - 1}$ and $-\frac{x^3 + 1}{x^3 - 1}$.
6. Insert 6 arithmetical means between 7 and 35.
7. Insert 12 arithmetical means between 37 and -28.

8. Insert 9 arithmetical means between $\frac{1}{3}$ and 12.

9. Insert 5 arithmetical means between $17\frac{2}{3}$ and $1\frac{1}{3}$.

10. Insert 20 arithmetical means between -16 and 26.

11. Insert 6 arithmetical means between $a+b$ and $8a-13b$.

Problems.

13. Pr. **1.** The sum of four numbers in arithmetical progression is 16, and their product is 105. What are the numbers?

We can express the four required numbers in terms of *two unknown* numbers.

Let $x-3d$, $x-d$, $x+d$, $x+3d$ be the four required numbers. Then, by the first condition,

$$(x-3d)+(x-d)+(x+d)+(x+3d)=16;$$

whence $x=4$.

By the second condition,

$$(x-3d)(x-d)(x+d)(x+3d)=105,$$

or $(x^2-9d^2)(x^2-d^2)=105.$

Substituting 4 for x and reducing, $9d^4-160d^2=-151$.

From this equation we obtain $d=\pm 1$, and $\pm\frac{1}{3}\sqrt{151}$.

The corresponding numbers are,

when $d=1$: 1, 3, 5, 7; when $d=-1$: 7, 5, 3, 1;

when $d=\frac{1}{3}\sqrt{151}$:

$$4-\sqrt{151},\ 4-\tfrac{1}{3}\sqrt{151},\ 4+\tfrac{1}{3}\sqrt{151},\ 4+\sqrt{151};$$

when $d=-\frac{1}{3}\sqrt{151}$:

$$4+\sqrt{151},\ 4+\tfrac{1}{3}\sqrt{151},\ 4-\tfrac{1}{3}\sqrt{151},\ 4-\sqrt{151}.$$

Notice the advantage of assuming the required numbers as in the above example. Had we assumed x, $x+d$, $x+2d$, $x+3d$ as the required numbers, the solution would have involved an equation of the fourth degree which could not have been solved as a quadratic.

Pr. **2.** Find the sum of all the numbers of three digits which are multiples of 7.

The numbers of three digits which are multiples of 7 are

$$7\times 15,\ 7\times 16,\ 7\times 17,\ \cdots,\ 7\times 142.$$

Their sum is $7(15 + 16 + \cdots + 142)$.

The series within the parentheses is an arithmetical progression, in which $a_1 = 15$, $d = 1$, $n = 128$, and $a_{128} = 142$.

Therefore $S_{128} = 10{,}048$.

The required sum is therefore $7 \times 10{,}048, = 70{,}336$.

EXERCISES III.

1. Find the sixth term, and the sum of eleven terms, of an A. P. whose eighth term is 11 and whose fourth term is -1.

2. The sixteenth term of an A. P. is -5, and the forty-first term is 45. What is the first term, and the sum of twenty terms?

3. Find the sum of all the even numbers from 2 to 50 inclusive.

4. Find the sum of thirty consecutive odd numbers, of which the last is 127.

5. The sum of the eighth and fourth terms of an A. P. of twenty terms is 24, and the sum of the fifteenth and nineteenth terms is 68. What are the elements of the progression?

6. The product of the first and fifth terms of an A. P. of ten terms is 85, and the product of the first and third terms is 55. What are the elements of the progression?

7. The first term of an A. P. of thirty terms is 100, and the sum of the first six terms is five times the sum of the six following terms. What are the elements of the progression?

8. The sum of the second and twentieth terms of an A. P. is 10, and their product is $23\frac{47}{64}$. What is the sum of sixteen terms?

9. The sixth term of an A. P. is 30, and the sum of the first thirteen terms is 455. What is the sum of the first thirty terms?

10. What value of x will make the arithmetical mean between $x^{\frac{1}{2}}$ and $x^{\frac{1}{4}}$ equal to 6?

11. Find the sum of all even numbers of two digits.

12. How many consecutive odd numbers beginning with 7 must be taken to give a sum 775?

13. Insert between 0 and 6 a number of arithmetical means so that the sum of the terms of the resulting A. P. shall be 39.

14. Find the number of arithmetical means between 1 and 19, if the first mean is to the last mean as 1 to 7.

15. The sum of the terms of an A. P. of six terms is 66, and the sum of the squares of the terms is 1006. What are the elements of the progression?

16. The sum of the terms of an A. P. of twelve terms is 354, and the sum of the even terms is to the sum of the odd terms as 32 to 27. What is the common difference?

17. How many positive integers of three digits are there which are divisible by 9? Find their sum.

18. If the sum of m terms of an A. P. is n, and the sum of n terms is m, what is the sum of $m + n$ terms? Of $m - n$ terms?

19. Show that the sum of $2n + 1$ consecutive integers is divisible by $2n + 1$.

20. Prove that if the same number be added to each term of an A. P., the resulting series will be an A. P.

21. Prove that if each term of an A. P. be multiplied by the same number, the resulting series will be an A. P.

22. Prove that if in the equation $y = ax + b$, we substitute c, $c + d$, $c + 2d$, $\cdots$, in turn for x, the resulting values of y will form an A. P.

23. Prove that if a^2, b^2, c^2 form an A. P., then

$$\frac{1}{b+c},\ \frac{1}{c+a},\ \frac{1}{a+b}\ \text{form an A. P.}$$

24. If a, b, c form an A. P., then

$$\tfrac{2}{9}(a+b+c)^3 = a^2(b+c) + b^2(a+c) + c^2(a+b).$$

25. A laborer agreed to dig a well on the following conditions: for the first yard he was to receive $2, for the second $2.50, for the third $3, and so on. If he received $42.50 for his work, how deep was the well?

26. On a certain day the temperature rose ½° hourly from 5 to 11 A.M., and the average temperature for that period was 8°. What was the temperature at 8 A.M.?

27. Twenty-five trees are planted in a straight line at intervals of 5 feet. To water them, the gardener must bring water for each tree separately from a well which is 10 feet from the first tree and in line with the trees. How far has the gardener walked when he has watered all the trees?

28. Two bodies, A and B, start at the same time from two points, C and D, which are 75 feet apart, and move in the same direction. A moves 1 foot the first second, 3 feet the second, 5 feet the third, etc.; B moves 3 feet the first second, 4 feet the second, 5 feet the third, etc. How long will it take A to overtake B?

29. A number of equal balls are placed in the form of a solid equilateral triangle in the following way: one ball is placed at the vertex, under this are placed two balls, under these two are placed three balls, and so on. If the number of balls is increased by 4, they can be placed in the form of a solid rectangle whose base is equal to the base of the triangle, and whose altitude is 3 balls shorter than the base. How many balls are in the triangle?

§ 3. GEOMETRICAL PROGRESSION.

1. A **Geometrical Series,** or as it is more commonly called a **Geometrical Progression** (G. P.), is a series in which each term after the first is formed by multiplying the preceding term by a constant number. See § 1, Art. 1, (2).

Evidently this definition is equivalent to the statement that the ratio of any term to the preceding is constant.

For this reason the constant multiplier of the first definition is called the **Ratio** of the progression.

Let a_1 stand for the first term of the series,
a_n for the nth (*any*) term,
r for the ratio,
and S_n for the sum of n terms.

The five numbers a_1, a_n, r, S_n, n, are called the **Elements** of the progression.

2. The ratio may be either larger or smaller than 1; in the former case the progression is called a *rising* or *ascending* progression; in the latter a *falling* or *descending* progression.

E.g., $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$, in which $r = \frac{3}{2}$,
and $\frac{1}{2} - 1 + 2 - 4 + 8 \cdots$, in which $r = -2$,
are ascending progressions; while
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, in which $r = \frac{1}{2}$,
and $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$, in which $r = -\frac{2}{3}$,
are *descending* progressions.

If the terms are all positive, the words *increasing* and *decreasing* may be used for ascending and descending, respectively.

3. *In a geometrical progression any term is equal to the first term multiplied by a power of the ratio whose exponent is one less than the number of the required term,* i.e.,

$$a_n = a_1 r^{n-1}. \qquad \text{(I.)}$$

By the definition of a geometrical progression

$$a_1 = a_1,\ a_2 = a_1 r,\ a_3 = a_2 r = a_1 r^2,\ a_4 = a_3 r = a_1 r^3, \text{ etc.}$$

The law expressed by the relations for these first four terms is evidently general, and since the exponent of r is one less than the number of the corresponding term, we have

$$a_n = a_1 r^{n-1}.$$

Ex. 1. If $a_1 = \frac{1}{2}$, $r = 3$, $n = 5$, then $a_5 = \frac{1}{2} \cdot 3^4 = \frac{81}{2}$.

This relation may also be used to find not only a_n when a_1, r, and n are given, but also to find the value of any one of the four numbers when the other three are given.

Ex. 2. If $a_1 = 4$, $a_6 = \frac{1}{8}$, $n = 6$, then $\frac{1}{8} = 4r^5$, whence $r = \frac{1}{2}$.

It is important to notice that, while a_1, a_n, and r may be positive or negative, integral or fractional, n must be a positive integer. Consequently a_1, a_n, r cannot be assumed arbitrarily.

As yet the value of n can be determined from (I.) only by inspection.

4. *In a geometrical progression*

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}, \qquad \text{(II.)}$$

or

$$S_n = \frac{a_1 - a_n r}{1 - r} = \frac{a_n r - a_1}{r - 1}. \qquad \text{(III.)}$$

We have $S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$, (1)

and $rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1} + a_1 r^n$. (2)

Consequently, subtracting (2) from (1),

$$S_n(1 - r) = a_1 - a_1 r^n,$$

whence

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}.$$

Substituting a_n for $a_1 r^{n-1}$ in (II.), we have

$$S_n = \frac{a_1 - a_n r}{1 - r} = \frac{a_n r - a_1}{r - 1}.$$

The first forms of (II.) and (III.) are to be used when $r < 1$, the second when $r > 1$.

5. Ex. 1. Given $a_1 = 3$, $r = 2$, $n = 6$, to find S_6.

From (II.), $$S_6 = \frac{3(2^6 - 1)}{2 - 1} = 189.$$

Formulæ (II.) and (III.) may be used not only to find S_n when a_1, r, and n, or a_1, a_n, and r are given, but also to find the value of any one of the four elements when the other three are given.

Ex. 2. Given $S_n = -63\frac{1}{2}$, $a_1 = -\frac{1}{2}$, $a_n = -32$, to find r.

By (III.), $-63\frac{1}{2} = \dfrac{-\frac{1}{2} + 32r}{1 - r}$, whence $r = 2$.

6. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five elements, a_1, a_n, r, S_n, n, when the three other elements are given.

Ex. 1. Given $r = -\frac{1}{2}$, $n = 5$, $a_5 = -\frac{1}{4}$, to find a_1 and S_5.

From (I.), $-\frac{1}{4} = a_1(-\frac{1}{2})^4$, whence $a_1 = -4$;

and from (II.), using the value of a_1 just found,

$$S_5 = \frac{-4\,[1-(-\frac{1}{2})^5]}{1-(-\frac{1}{2})} = -\frac{11}{4}.$$

Ex. 2. Given $r = 2$, $a_n = 16$, $S_n = 31\frac{1}{2}$, to find a_1 and n.

From (III.), $31\frac{1}{2} = \dfrac{16 \times 2 - a_1}{2 - 1}$, whence $a_1 = \frac{1}{2}$.

From (I.), $16 = \frac{1}{2} \cdot 2^{n-1}$, whence $n = 6$.

Ex. 3. Given $n = 7$, $a_7 = 16$, $S_7 = 31\frac{3}{4}$, to find r and a_1.

From (I.), $16 = a_1 r^6$,

and from (II.), $31\frac{3}{4} = \dfrac{a_1(1 - r^7)}{1 - r}.$

Eliminating a_1 between these two equations, we obtain

$$63\,r^7 - 127\,r^6 = -64.$$

Thus this example leads to an equation of the seventh degree, which cannot be solved. The value of r in such equations can often be found by inspection. In the above equation $r = 2$. We then have $a_1 = \frac{1}{4}$.

7. In many examples the elements necessary for determining the element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. 1. Given $a_5 = 48$, $a_8 = 384$, to find a_1 and r.

From (I.), $48 = a_1r^4$, and $384 = a_1r^7$;

whence $r^3 = 8$, or $r = 2$. Therefore $a_1 = 3$.

Or, we could have regarded 48 as the first term and 384 as the last term of a progression of four terms. Then by (I.), $384 = 48\,r^3$, whence $r = 2$, as before.

Ex. 2. Given $S_6 = 63$, $S_9 = 511$, to find a_1 and r.

$$\text{From (II.),}\quad 63 = \frac{a_1(r^6-1)}{r-1},\ \text{and}\ 511 = \frac{a_1(r^9-1)}{r-1}; \qquad (1)$$

$$\text{whence}\quad \frac{r^9-1}{r^6-1} = \frac{511}{63},\ \text{or}\ \frac{r^6+r^3+1}{r^3+1} = \frac{511}{63}.$$

Therefore $63\,r^6 - 448\,r^3 = 448$;

whence $r^3 = 8$, and $-\frac{8}{9}$; and $r = 2$, and $-\frac{2}{3}\sqrt[3]{3}$.

We then have from (1): $a_1 = 1$, when $r = 2$;

and $a_1 = 100\frac{1}{17}(2\sqrt[3]{3} + 3)$, when $r = -\frac{2}{3}\sqrt[3]{3}$.

Such examples as the last in general lead to equations of higher degree than the second.

EXERCISES IV.

Find the last term and the sum of the terms of each of the following geometrical progressions:

1. $3 + 6 + \cdots$ to 6 terms.
2. $2 - 4 + \cdots$ to 10 terms.
3. $32 - 16 + \cdots$ to 7 terms.
4. $1\frac{3}{5} + 2\frac{2}{3} + \cdots$ to 6 terms.
5. $2 - 2^2 + \cdots$ to 11 terms.
6. $\frac{1}{\sqrt{2}} + \frac{1}{2} + \cdots$ to n terms.
7. $1 + (1 + a) + \cdots$ to 4 terms, to n terms.
8. $a^p + a^{p+q} + \cdots$ to 7 terms, to n terms.

In each of the following geometrical progressions find the values of the elements not given:

9. $a_1 = 1$, $r = 4$, $n = 5$.
10. $a_1 = 2\frac{1}{3}$, $r = -2$, $n = 6$.
11. $a_n = 10$, $r = 2$, $n = 4$.
12. $a_n = 1.2$, $r = -.2$, $n = 5$.
13. $r = 2$, $n = 5$, $S_n = 62$.
14. $r = 10$, $n = 7$, $S_n = 3{,}333{,}333$.
15. $a_1 = 5$, $n = 9$, $a_n = 327{,}680$.
16. $a_1 = 74\frac{2}{3}$, $n = 6$, $a_n = 2\frac{1}{3}$.
17. $a_n = 96$, $n = 4$, $S_n = 127.5$.
18. $a_n = 7$, $n = 9$, $S_n = 68{,}887$.

19. $a_1 = 1,\ r = 2,\ a_n = 64.$ **20.** $a_1 = 7,\ r = 10,\ a_n = 700.$

21. $a_1 = 74\frac{2}{3},\ a_n = 2\frac{1}{3},\ S_n = 147.$ **22.** $a_1 = 1,\ a_n = 512,\ S_n = 1023.$

23. $a_n = 44,\ r = 4,\ S_n = 55.$ **24.** $a_n = 3125,\ r = 5,\ S_n = 3905.$

25. $a_1 = 40,\ r = \frac{1}{2},\ S_n = 75.$ **26.** $a_1 = 4,\ r = 3,\ S_n = 118{,}096.$

27. $a_1 = 3,\ n = 3,\ S_n = 1953.$ **28.** $a_1 = 100,\ n = 3,\ S_n = 700.$

Sum of an Infinite Geometrical Progression.

8. When r is less than 1, the term a_1r^n in the formula

$$S = \frac{a_1 - a_1r^n}{1 - r}$$

decreases as n increases. As n grows indefinitely large, a_1r^n becomes indefinitely small. For, as in Ch. XVII., Art. 15, we have

$$a^{n+1} > b^{n+1} + (n+1)(a-b)b^n.$$

Now let $a = 1$, $b = 1 - d$, wherein d is a positive proper fraction. Then

$$1 > (1-d)^{n+1} + (n+1)d(1-d)^n;\ \text{or,}\ 1 > (1-d)^n(1+nd).$$

Therefore $$(1-d)^n < \frac{1}{1+nd}.$$

Evidently, $\frac{1}{1+nd}$ can be made less than any assigned number, however small, by increasing n indefinitely. Therefore, $(1-d)^n$, which may represent *any proper fraction*, say r, can be made less than any assigned number, however small. That is, when $n = \infty$, $a_1r^n = 0$.

We therefore have, when $r < 1$ and $n = \infty$,

$$S_\infty = \frac{a_1}{1 - r}.$$

Ex. 1. Find the sum of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots,\ \text{in which}\ r = \tfrac{1}{2}.$$

We have $$S_\infty = \frac{1}{1 - \frac{1}{2}} = 2.$$

The meaning of this result is that the sum of the given series approaches the finite value 2 more and more nearly as more and more terms are included in the sum, and that the sum can be made to differ from 2 by as little as we please, by taking a sufficient number of terms. The *exact* sum 2, however, can never be obtained.

This theory can be applied to find the value of a repeating (recurring) decimal.

Ex. 2. Verify that $.\dot{6} = \frac{2}{3}$.

We have $.666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots,$

a geometrical progression whose first term is $\frac{6}{10}$ and whose ratio is $\frac{1}{10}$. Consequently

$$S_\infty = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{9} = \frac{2}{3}.$$

EXERCISES V.

Find the sum of the following infinite geometrical progressions:

1. $6 + 4 + \cdots.$ 2. $60 + 15 + \cdots.$ 3. $10 - 6 + \cdots.$

4. $\frac{1}{2} + \frac{1}{4} + \cdots.$ 5. $1 - \frac{1}{3} + \cdots.$ 6. $5 - \frac{1}{2} + \cdots.$

7. $\frac{3}{2} - \frac{2}{3} + \cdots.$ 8. $\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} + \cdots.$ 9. $\sqrt{.2} + \sqrt{\frac{1}{125}} + \cdots.$

10. $1 + x + x^2 + \cdots$, when $x < 1$. 11. $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$, when $x > 1$.

Find the value of each of the following repeating decimals:

12. $.44 \cdots.$ 13. $.99 \cdots.$ 14. $.2727 \cdots.$ 15. $.015015 \cdots.$

16. $.199 \cdots.$ 17. $1.0909 \cdots.$ 18. $.122323 \cdots.$ 19. $.201475475 \cdots.$

Verify each of the following identities:

20. $\sqrt{.44 \cdots} = .66 \cdots.$ 21. $\sqrt{.6944 \cdots} = .833 \cdots.$

Geometrical Means.

9. A **Geometrical Mean** between two numbers is a number, in value between the two, which forms with them a geometrical progression.

E.g., $+2$, or -2, is a geometrical mean between 1 and 4.

Let G be the geometrical mean between a and b.

Then by definition of a geometrical progression,

$$\frac{G}{a} = \frac{b}{G}; \text{ whence } G = \pm\sqrt{(ab)}.$$

That is, the geometrical mean between two numbers is the square root of their product.

Ex. Find the geometrical mean between 1 and $\frac{4}{9}$. We have

$$G = \pm\sqrt{(1 \times \tfrac{4}{9})} = \pm \tfrac{2}{3}.$$

10. **Geometrical Means** between two numbers are numbers, in value between the two, which form with them a geometrical progression. *E.g.*, 4 and 16 are two geometrical means between

1 and 64; and 2, 4, 8, 16, 32 are five geometrical means between 1 and 64.

Ex. Insert five geometrical means between 2 and 11.

We have $11 = 2r^6$, whence $r = \sqrt[6]{\frac{11}{2}}$.

The required means are

$$2\sqrt[6]{\tfrac{11}{2}},\ 2\sqrt[6]{(\tfrac{11}{2})^2},\ 2\sqrt[6]{(\tfrac{11}{2})^3},\ 2\sqrt[6]{(\tfrac{11}{2})^4},\ 2\sqrt[6]{(\tfrac{11}{2})^5}.$$

11. If n geometrical means be inserted between a and b, we have a geometrical progression of $n+2$ terms.

Consequently, by (I.),

$$b = ar^{n+1}, \text{ or } r = \sqrt[n+1]{\frac{b}{a}}.$$

The progression is therefore,

$$a,\ a \times \sqrt[n+1]{\frac{b}{a}},\ a \times \sqrt[n+1]{\left(\frac{b}{a}\right)^2},\ a \times \sqrt[n+1]{\left(\frac{b}{a}\right)^3},\ \cdots b,$$

or

$$a,\ \sqrt[n+1]{(a^n b)},\ \sqrt[n+1]{(a^{n-1}b^2)},\ \sqrt[n+1]{(a^{n-2}b^3)},\ \cdots b.$$

EXERCISES VI.

Insert a geometrical mean between

1. 2 and 8.
2. 12 and 3.
3. $\frac{1}{5}$ and $\frac{1}{125}$.
4. $\sqrt{a}$ and $\sqrt{(2a)}$.
5. $75m^3$ and $3mn^4$.
6. $\frac{p}{q}$ and $\frac{q}{p}$.
7. $(a-b)^2$ and $(a+b)^2$.
8. $(a^2+1)(a^2-1)^{-1}$ and $\frac{1}{4}(a^4-1)$.
9. Insert 5 geometrical means between 2 and 1458.
10. Insert 7 geometrical means between 2 and 512.
11. Insert 6 geometrical means between 3 and -384.
12. Insert 6 geometrical means between 5 and -640.
13. Insert 8 geometrical means between 4 and $-\frac{19683}{128}$.
14. Insert 9 geometrical means between 1 and $\frac{1024}{59049}$.

Problems.

12. Pr. The sum of the terms of a geometrical progression of five terms is 484; the sum of the second and fourth terms is 120. What is the progression?

Let $\frac{x}{r^2}, \frac{x}{r}, x, xr, xr^2$ be the required terms.

By the first condition, $\frac{x}{r^2}+\frac{x}{r}+x+xr+xr^2=484.$ (1)

By the second condition, $\frac{x}{r}+xr=120.$ (2)

Subtracting (2) from (1), $\frac{x}{r^2}+x+xr^2=364.$ (3)

Dividing by x, and adding 1 to both members,

$$\frac{1}{r^2}+2+r^2=\frac{364}{x}+1. \qquad (4)$$

Squaring (2), $x^2\left(\frac{1}{r^2}+2+r^2\right)=14400,$ (5)

or
$$\frac{1}{r^2}+2+r^2=\frac{14400}{x^2}. \qquad (6)$$

Equating second members of (4) and (6),

$$\frac{364}{x}+1=\frac{14400}{x^2}, \qquad (7)$$

whence $x=36$, and -400.

Substituting 36 for x in (2), we obtain

$$r=3, \text{ and } \tfrac{1}{3}.$$

The value, -400, of x gives imaginary values of r, and therefore must be rejected.

When $x=36$ and $r=3$, the series is 4, 12, 36, 108, 324· when $x=36$ and $r=\frac{1}{3}$, the series is 324, 108, 36, 12, 4.

EXERCISES VII.

1. The first term of a G. P. of six terms is 768, and the last term is one-sixteenth of the fourth term. What is the sum of the six terms of the progression?

2. The first term of a G. P. of ten terms is 3, and the sum of the first three terms is one-eighth of the sum of the next three terms. Find the elements of the progression.

3. The twelfth term of a G. P. is 1536, and the fourth term is 6. What is the ratio, and the sum of the first eleven terms?

4. The sum of the first and fourth terms of a G. P. of ten terms is 455, and the sum of the second and third terms is 140. What are the elements of the progression?

5. In a G. P. of eight terms, the sum of the first seven terms is $444\frac{1}{2}$, and is to the sum of the last seven terms as 1 to 2. Find the elements of the progression.

6. The sum of the first four terms of a G. P. is 15, and the sum of the terms from the second to the fifth inclusive is 30. What is the first term, and the ratio?

7. Find the elements of a G. P. of six terms whose first term is 1, and the sum of whose first six terms is twenty-eight times the sum of the first three terms.

8. The sum of the first three terms of a G. P. is 21, and the sum of their squares is 189. What is the first term?

9. The product of the first and third terms of a G. P. of seven terms is 64, and the sum of the fourth and sixth terms is 6. What are the elements of the progression?

10. The product of the first three terms of a G. P. is 216, and the sum of their cubes is 1971. What is the first term, and the ratio?

11. Find the mth and the nth terms of a geometrical progression whose $(m+n)$th term is k, and $(m-n)$th term is l.

12. If the numbers 1, 1, 3, 9 be added to the first four terms of an A. P., respectively, the resulting terms will form a G. P. What is the first term, and the common difference of the A. P.?

13. A G. P. and an A. P. have a common first term 3, the difference between their second terms is 6, and their third terms are equal. What is the ratio of the G. P., and the common difference of the A. P.?

14. The sum of the eight terms of an A. P., whose first term is 1, is 3,294,176. The first and last terms of a G. P. of eight terms are equal to the corresponding terms of the A. P. Find the fifth term of the G. P.

15. The first and last terms of an A. P. of fifteen terms are equal to the corresponding terms of a G. P. of fifteen terms, and the ninth term of the A. P. is equal to the eighth term of the G. P. What is the ratio of the G. P.?

16. The ratios of two geometrical progressions having the same number of terms are $\frac{3}{10}$ and $\frac{1}{5}$ respectively, and their first terms are equal. The last term of the first progression is 243, and the last term of the second is 32. Find the elements of each progression.

17. Show that, if all the terms of a G. P. be multiplied by the same number, the resulting series will form a G. P.

18. Show that the series whose terms are the reciprocals of the terms of a G. P. is a G. P.

19. Show that the product of the first and last terms of a G. P. is equal to the product of any two terms which are equally distant from the first and last terms respectively.

20. If the numbers a, b, c, d form a G. P., show that

$$(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2.$$

21. A merchant's investment yields him each year after the first, three times as much as the preceding year. If his investment paid him $ 9720 in four years, how much did he realize the first year and the fourth year?

22. On one of the sides of an acute angle a point is taken a feet from the vertex; from this point a perpendicular is let fall on the second side, cutting off b feet from the vertex. From the foot of this perpendicular a perpendicular is let fall on the first side, and from the foot of this perpendicular a third perpendicular is let fall on the second side, and so on indefinitely. Find the sum of all the perpendiculars.

23. Given a square whose side is $2a$. The middle points of its adjacent sides are joined by lines forming a second square inscribed in the first. In the same manner a third square is inscribed in the second, a fourth in the third, and so on indefinitely. Find the sum of the perimeters of all the squares.

§ 4. HARMONICAL PROGRESSION.

1. A **Harmonical Progression** (H. P.) is a series the reciprocals of whose terms form an arithmetical progression.

E.g.,

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$

is a harmonical progression, since

$$1+2+3+4+\cdots$$

is an arithmetical progression.

Consequently to every harmonical progression there corresponds an arithmetical progression, and *vice versa.*

2. *If three numbers be in harmonical progression, the ratio of the difference between the first and second terms to the difference between the second and third terms is equal to the ratio of the first term to the third term.*

Let the three numbers a, b, c be in harmonical progression.

By the definition of a harmonical progression,

$$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$$

are in arithmetical progression. Consequently

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},\ \text{or}\ \frac{a-b}{ab}=\frac{b-c}{bc},\ \text{whence}\ \frac{a-b}{b-c}=\frac{a}{c}.$$

3. Any term of a harmonical progression is obtained by finding the same term of the corresponding arithmetical progression and taking its reciprocal.

Ex. Find the eleventh term of the harmonical progression $4, 2, \frac{4}{3}, \cdots$.

The corresponding arithmetical progression is

$$\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \cdots,$$

and its eleventh term is $\frac{11}{4}$.

Therefore the eleventh term of the given progression is $\frac{4}{11}$.

4. No formula has been derived for the sum of n terms of a harmonical progression.

5. A **Harmonical Mean** between two numbers is a number, in value between the two, which forms with them a harmonical progression.

E.g., $\frac{3}{2}$ is a harmonical mean between $\frac{1}{2}$ and $-\frac{3}{2}$.

Let H stand for the harmonical mean between a and b, then $\frac{1}{H}$ is an arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$. Consequently

$$\frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2}, \text{ or } H = \frac{2ab}{a+b}.$$

Ex. Insert a harmonical mean between 2 and 5.

We have $$H = \frac{2 \times 2 \times 5}{2+5} = \frac{20}{7}.$$

6. **Harmonical Means** between two numbers are numbers, in value between the two, which form with them a harmonical progression.

E.g., $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}$ are five harmonical means between 3 and $\frac{3}{7}$.

Ex. Insert four harmonical means between 1 and 10.

We have first to insert four arithmetical means between 1 and $\frac{1}{10}$, and obtain

$$\tfrac{41}{50}, \tfrac{32}{50}, \tfrac{23}{50}, \tfrac{14}{50}.$$

The required harmonical means are therefore

$$\tfrac{50}{41}, \tfrac{50}{32}, \tfrac{50}{23}, \tfrac{50}{14}.$$

7. To insert n harmonical means between a and b, we insert n arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and take their reciprocals. The n arithmetical means are

$$\frac{n\frac{1}{a}+\frac{1}{b}}{n+1},\quad \frac{(n-1)\frac{1}{a}+2\frac{1}{b}}{n+1},\quad \frac{(n-2)\frac{1}{a}+3\frac{1}{b}}{n+1},\ \cdots.$$

or
$$\frac{a+nb}{(n+1)ab},\quad \frac{2a+(n-1)b}{(n+1)ab},\quad \frac{3a+(n-2)b}{(n+1)ab},\ \cdots$$

Consequently the required harmonical means are

$$\frac{(n+1)ab}{a+nb},\quad \frac{(n+1)ab}{2a+(n-1)b},\quad \frac{(n+1)ab}{3a+(n-2)b},\ \cdots.$$

Problems.

8. Pr. 1. The geometrical mean between two numbers is $\frac{1}{2}$, and the harmonical mean is $\frac{2}{5}$. What are the numbers?

Let x and y represent the two numbers.

Then $$\sqrt{(xy)}=\tfrac{1}{2},\ \text{or}\ xy=\tfrac{1}{4}; \qquad (1)$$

and $$\frac{2xy}{x+y}=\frac{2}{5},\ \text{or}\ 5xy=x+y. \qquad (2)$$

Solving (1) and (2), we obtain $x=1$, $y=\frac{1}{4}$, and $x=\frac{1}{4}$, $y=1$.

Pr. 2. If to each of three numbers in geometrical progression the second number be added, the resulting series will form a harmonical progression. What are the numbers?

Let $\frac{a}{r}$, a, ar represent the three numbers.

Then we are to prove that

$$\frac{a}{r}+a,\ 2a,\ ar+a$$

is a harmonical progression; that is, that

$$\frac{r}{a(1+r)},\ \frac{1}{2a},\ \frac{1}{a(1+r)}$$

is an arithmetical progression.

We have $$\frac{\frac{r}{a(1+r)}+\frac{1}{a(1+r)}}{2}=\frac{1}{2a}.$$

That is, $\frac{1}{2a}$ is an arithmetical mean between $\frac{r}{a(1+r)}$ and $\frac{1}{a(1+r)}$. Consequently, $\frac{a}{r}+a,\ 2a,\ ar+a$ is a harmonical progression.

EXERCISES VIII.

Find the last term of each of the following harmonical progressions:

1. $1+\frac{2}{3}+\frac{1}{2}+\cdots$ to 8 terms. **2.** $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\cdots$ to 15 terms.

3. $2-2-\frac{2}{3}-\cdots$ to 11 terms. **4.** $-8-\frac{8}{9}-\frac{8}{17}-\cdots$ to 16 terms.

5. $\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\cdots$ to 25 terms.

6. $\frac{1}{\sqrt{2}}+\frac{1}{1+\sqrt{2}}+\frac{1}{2+\sqrt{2}}+\cdots$ to 30 terms.

Find the harmonical mean between

7. 2 and 4. **8.** -3 and 4. **9.** $\frac{1}{7}$ and $\frac{1}{9}$.

10. $\frac{1}{x-1}$ and $-\frac{1}{x+1}$. **11.** $\frac{a-b}{a+b}$ and $\frac{a+b}{a-b}$.

12. Insert 5 harmonical means between 5 and $\frac{1}{5}$.

13. Insert 10 harmonical means between 3 and $\frac{1}{4}$.

14. Insert 4 harmonical means between -7 and $\frac{1}{2}$.

15. If x^2, y^2, z^2 be in A.P., prove that $y+z$, $z+x$, and $x+y$ are in H.P.

16. If y be the harmonical mean between x and z, prove that

$$\frac{1}{y-x}+\frac{1}{y-z}=\frac{1}{x}+\frac{1}{z}.$$

17. The arithmetical mean between two numbers is 6, and the harmonical mean is $\frac{35}{6}$. What are the numbers?

18. If one number exceeds another by two, and if the arithmetical mean exceeds the harmonical mean by $\frac{1}{10}$, what are the numbers?

19. The seventh term of a harmonical progression is $\frac{1}{15}$, and the twelfth term is $\frac{1}{25}$. What is the twentieth term?

20. The tenth term of a harmonical progression is $\frac{1}{5}$, and the twentieth term is $\frac{1}{10}$. What is the first term?

CHAPTER XXVIII.

THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS.

1. The expansions of the powers of a binomial, from the first to the sixth inclusive, were given in Ch. VI., § 1, Art. 8, and the laws governing the expansions of these powers were stated.

As yet, however, we cannot infer that these laws hold for the seventh power without multiplying the expansion of the sixth power by $a+b$; nor for the eighth power without next multiplying the expansion of the seventh power by $a+b$; and so on.

If, however, we prove that, provided the laws hold for any particular power, they hold for the next higher power, we can infer, without further proof, that because the laws hold for the sixth power, they hold also for the seventh; then that because they hold for the seventh, they hold also for the eighth, and so on to any higher power.

2. If the laws (i.)–(vi.) hold for the rth power, we have

$$(a+b)^r=a^r+ra^{r-1}b+\frac{r(r-1)}{1\cdot 2}a^{r-2}b^2+\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}a^{r-3}b^3+\cdots.$$

Notice that only the first four terms of the expansion are written. But it is often necessary to write any term (the kth, say) without having written all the preceding terms.

To derive this term, observe that the following laws hold for each term of the expansion:

(i.) *The exponent of b is one less than the number of the term (counting from the left).*

Thus, in the first term we have $b^{1-1}=b^0=1$; in the second, $b^{2-1}=b$; in the tenth, $b^{10-1}=b^9$; and in the kth term, b^{k-1}.

(ii.) *The exponent of a is equal to the binomial exponent less the exponent of b.*

Thus, in the first term we have $a^{r-0} = a^r$; in the second, a^{r-1}; in the tenth, a^{r-9}; and in the kth term, $a^{r-(k-1)}, = a^{r-k+1}$.

(iii.) *The number of factors (beginning with* 1 *and increasing by* 1) *in the denominator of each coefficient, and the number of factors (beginning with r and decreasing by* 1) *in the numerator of each coefficient, is equal to the exponent of b in that term.*

Thus, in the coefficient of the second term the denominator is 1 and the numerator is r; in that of the second term the denominator is $1 \cdot 2$ and the numerator is $r(r-1)$; in the tenth term the denominator is $1 \cdot 2 \cdots 9$ and the numerator is $r(r-1) \cdots (r-8)$; and in the kth term the denominator is $1 \cdot 2 \cdot 3 \cdots (k-1)$, and the numerator is

$$r(r-1) \cdots [r-(k-2)], = r(r-1) \cdots (r-k+2).$$

Therefore the kth term in the expansion of $(a+b)^r$ is

$$\frac{r(r-1)(r-2) \cdots (r-k+2)}{1 \cdot 2 \cdot 3 \cdots (k-1)} a^{r-k+1} b^{k-1}.$$

In like manner, any other term can be written.
Thus, the $(k-1)$th term is

$$\frac{r(r-1)(r-2) \cdots (r-k+3)}{1 \cdot 2 \cdot 3 \cdots (k-2)} a^{r-k+2} b^{k-2}.$$

3. We can now prove that, if the laws (i.)–(vi.) hold for $(a+b)^r$, they also hold for $(a+b)^{r+1}$; that is, if they hold for any power they hold for the next higher power. Assuming, then, that the laws hold for $(a+b)^r$, we have

$$(a+b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{1 \cdot 2} a^{r-2}b^2 + \cdots$$
$$+ \frac{r(r-1)(r-2) \cdots (r-k+3)}{1 \cdot 2 \cdot 3 \cdots (k-2)} a^{r-k+2} b^{k-2}$$
$$+ \frac{r(r-1)(r-2) \cdots (r-k+3)(r-k+2)}{1 \cdot 2 \cdot 3 \cdots (k-2)(k-1)} a^{r-k+1} b^{k-1} + \cdots.$$

The first three terms of the expansion are written, then all terms are omitted, except the $(k-1)$th and the kth.

Multiplying the expansion of $(a+b)^r$ by $(a+b)$, we obtain

$$(a+b)^{r+1}=$$

$$a^{r+1}+ra^rb+\frac{r(r-1)}{1\cdot 2}a^{r-1}b^2+\cdots+\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1}+\cdots$$

$$+a^rb+ra^{r-1}b^2+\cdots+\frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}a^{r-k+2}b^{k-1}+\cdots$$

$$=a^{r+1}+(r+1)a^rb+\left[\frac{r(r-1)}{1\cdot 2}+r\right]a^{r-1}b^2+\cdots$$

$$+\left[\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)}+\frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}\right]a^{r-k+2}b^{k-1}+\cdots.$$

But
$$\frac{r(r-1)}{1\cdot 2}+r=\frac{r^2-r+2r}{1\cdot 2}=\frac{(r+1)r}{1\cdot 2};$$

and
$$\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)}+\frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}$$

$$=\frac{r(r-1)\cdots(r-k+2)+r(r-1)\cdots(r-k+3)(k-1)}{1\cdot 2\cdots(k-1)}$$

$$=\frac{r(r-1)\cdots(r-k+3)(r-k+2+k-1)}{1\cdot 2\cdots(k-1)}$$

$$=\frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}.$$

Therefore,

$$(a+b)^{r+1}=a^{r+1}+(r+1)a^rb+\frac{(r+1)r}{1\cdot 2}a^{r-1}b^2+\cdots$$

$$+\frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1}+\cdots.$$

The laws (i.)–(vi.) hold for the above expansion of $(a+b)^{r+1}$. We therefore conclude that if the expansion holds for $(a+b)^r$, it also holds for $(a+b)^{r+1}$.

Consequently, since the expansion holds for the sixth power, it holds for the seventh, and so on to any positive integral power.

The method of proof employed in this article is called **Proof by Mathematical Induction.**

4. We may now write the expansion of $(a+b)^n$, wherein n is any positive integer:

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\cdots.$$

In particular, if $a=1$,

$$(1+b)^n=1+nb+\frac{n(n-1)}{1\cdot 2}b^2+\cdots.$$

5. The expansion of $(a-b)^n$ can be at once written from that of $(a+b)^n$.

We have $$(a-b)^n = [a+(-b)]^n$$

$$= a^n + na^{n-1}(-b) + \frac{n(n-1)}{1\cdot 2}a^{n-2}(-b)^2 + \cdots$$

$$= a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \cdots.$$

Observe that the signs of the terms alternate, + and −, beginning with the first, or that the terms containing *even* powers of b are *positive*, and those containing *odd* powers of b are *negative*.

6. Ex. 1. Find the seventh term in $(2x+3y)^{11}$.

In the seventh term the exponent of $3y(=b)$ is 6; the exponent of $2x(=a)$ is $11-6, =5$. The denominator of the coefficient contains six factors beginning with 1, and the numerator contains six factors beginning with 11. Therefore the seventh term is

$$\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(2x)^5(3y)^6, \ = 10777536\,x^5y^6.$$

Ex. 2. Find the first *five* terms of $(a^{-\frac{1}{2}} - 2b^{-2})^{11}$.

We have $(a^{-\frac{1}{2}} - 2b^{-2})^{11}$

$$= (a^{-\frac{1}{2}})^{11} - 11(a^{-\frac{1}{2}})^{10}(2b^{-2}) + \frac{11\cdot 10}{1\cdot 2}(a^{-\frac{1}{2}})^9(2b^{-2})^2$$

$$- \frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3}(a^{-\frac{1}{2}})^8(2b^{-2})^3 + \frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4}(a^{-\frac{1}{2}})^7(2b^{-2})^4 - \cdots$$

$$= a^{-\frac{11}{2}} - 22\,a^{-5}b^{-2} + 220\,a^{-\frac{9}{2}}b^{-4} - 1320\,a^{-4}b^{-6} + 5280\,a^{-\frac{7}{2}}b^{-8} - \cdots.$$

Ex. 3. Write the term containing x^{10} in $\left(2x^2 - \frac{3}{x^{\frac{1}{2}}}\right)^{15}$.

Let k stand for the number of the required term. Then, neglecting the coefficient, we have $(2x^2)^{15-k+1}\left(\frac{3}{x^{\frac{1}{2}}}\right)^{k-1}$.

The power of x obtained from this expression is

$$(x^2)^{15-k+1}(x^{-\frac{1}{2}})^{k-1}, \ = x^{32\frac{1}{2}-2\frac{1}{2}k}.$$

In order that this power of x may be equal to x^{10}, we must have

$$32\tfrac{1}{2} - 2\tfrac{1}{2}\,k = 10;$$

whence $k = 9$, the number of the required term.

We now have

$$\begin{aligned}\text{ninth term} &= \frac{15 \cdot 14 \cdot 13 \cdot \cdots \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 7 \cdot 8}(2\,x^2)^7\left(\frac{3}{x^{\frac{1}{2}}}\right)^8 \\ &= 13 \cdot 11 \cdot 5 \cdot 9 \cdot 2^7 \cdot 3^8 x^{10}.\end{aligned}$$

7. The coefficients in the expansion of $(a + b)^n$ are called **Binomial Coefficients.** They may be represented by the following abbreviations:

$$n = \frac{n}{1} = \binom{n}{1},\quad \frac{n(n-1)}{1 \cdot 2} = \binom{n}{2},\quad \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \binom{n}{3},$$

$$\frac{n(n-1)\cdots(n-k+2)}{1 \cdot 2 \cdot 3 \cdots (k-1)} = \binom{n}{k-1}.$$

Observe that in the symbolic notation the upper number is the binomial exponent and the lower number is the number of factors in numerator and denominator.

E.g.,
$$\binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}.$$

The binomial expansion may now be written

$$\begin{aligned}(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \\ + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.\end{aligned}$$

The terms beginning at the right-hand end of the expansion are determined from the principle that the lower number in any binomial coefficient is equal to the exponent of b in that term.

EXERCISES.

Write the expansion of each of the following powers:

1. $(5\,ab - 3\,a^2)^4$.
2. $(a^{\frac{1}{2}} + 2\,a^2x^{-1})^4$.
3. $(x - \sqrt{-3\,y})^4$.
4. $(4\,a^{-2} - 5\,x)^5$.
5. $(\frac{1}{2}\,a^{\frac{3}{5}} + \frac{2}{3}\,x^3)^5$.
6. $(\sqrt{a} + \sqrt{b})^5$.
7. $(n - \frac{1}{2})^6$.
8. $(x^2 + 2\,a^{-2}x)^6$.
9. $(3\,n^{\frac{4}{3}} - \frac{2}{3}\,n^2y^{-2})^7$.
10. $\left(\dfrac{2}{\sqrt[3]{a^2}} - \dfrac{a\sqrt{a}}{2}\right)^5$.
11. $\left(n^2 + \dfrac{2\,a}{n^{-1}}\right)^6$.
12. $\left(\dfrac{a^{\frac{3}{2}}x^2}{n} - \dfrac{2\,n^{\frac{2}{3}}}{a^2x^{-1}}\right)^6$.

13. $(\sqrt{-2}+2x^{-\frac{2}{7}})^7$. **14.** $(a^2+x)^8$. **15.** $(\sqrt[4]{a}+\sqrt[4]{b})^8$.

16. $(a-\sqrt{-a})^8$. **17.** $(x^{-1}-x)^9$. **18.** $(ab^{-2}-b^2x)^9$.

19. $(x^2-\sqrt{-x})^9$. **20.** $(x^2-1)^{10}$. **21.** $(a^2b+b^{-3})^{10}$.

22. $\left(\frac{a^2b}{2x^{\frac{1}{2}}}+\frac{2x}{a^{-1}b^2}\right)^7$. **23.** $\left(\sqrt{\frac{a}{n}}+\sqrt{\frac{n}{a}}\right)^9$. **24.** $\left(x-\frac{a}{x}\right)^{10}$.

25. $[\sqrt{(x+1)}-\sqrt{(x-1)}]^4$. **26.** $[\sqrt[3]{(a+b)}+\sqrt[3]{(a-b)}]^6$.

Simplify each of the following expressions:

27. $(1+\sqrt{-x})^8+(1-\sqrt{-x})^8$. **28.** $(x+\sqrt{-3})^9-(x-\sqrt{-3})^9$.

Write the expansion of each of the following powers:

29. $(1-x+x^2)^8$. **30.** $(1+a^{\frac{1}{2}}-a^{-2})^8$.

31. $(2-3x+x^2)^4$. **32.** $(1-x\sqrt{2}+x^2\sqrt{3})^4$.

Write the

33. 3d term of $(a+b)^{15}$. **34.** 5th term of $(a-b)^{16}$.

35. 3d term of $(x^3+\frac{1}{3})^{18}$. **36.** 8th term of $(a^2-b^2)^{12}$.

37. 6th term of $(a^{\frac{1}{10}}+b^{\frac{1}{5}})^{15}$. **38.** 7th term of $(a^n-a^{-n})^{14}$.

39. 6th term of $\left(\sqrt[3]{m}-\frac{2x}{\sqrt[3]{m^2}}\right)^{12}$. **40.** 15th term of $\left(a^3+\frac{1}{a}\right)^{20}$.

41. 12th term of $(x-\sqrt{-x})^{20}$. **42.** 9th term of $(\sqrt{x}-ax^{\frac{2}{3}})^{16}$.

43. 9th term of $[1-\sqrt{(1-\sqrt{2})}]^{12}$.

44. Write the middle term of $(x\sqrt{x}-1)^4$.

45. Write the middle terms of $(a^{\frac{1}{5}}+x^{\frac{1}{2}})^9$.

46. Write the term of $\left(5x^3-\frac{3}{2x^2}\right)^9$ which contains x^{12}.

47. Write the term of $(2a^{-3}-27a^{\frac{3}{5}})^{15}$ which contains $a^{-30\frac{3}{5}}$.

CHAPTER XXIX.

PERMUTATIONS AND COMBINATIONS.

§ 1. DEFINITIONS.

1. The following examples will illustrate the character of an important class of problems.

Pr. **1.** Write the numbers of two figures each which can be formed from the three figures, 4, 5, 6.

We have 45, 54, 46, 64, 56, 65.

Pr. **2.** What committees of two persons each can be appointed from the three persons, A, B, C?

The committees may consist of A, B; A, C; or B, C.

These problems make clear the difference between groups of things, selected from a given number of things, in which *the order is taken into account*, as in Pr. 1, and in which *the order is not taken into account*, as in Pr. 2.

2. We are thus naturally led to the following definitions:

A Permutation of any number of things is a group of some or all of them, *arranged in a definite order.*

A Combination of any number of things is a group of some or all of them, *without reference to order.*

3. It follows from these definitions that two permutations are different when some or all of the things in them are different, or when their order of arrangement is different; and that two combinations are different only when at least one thing in one is not contained in the other.

Thus, *ab* and *ba* are different permutations, but the same combination.

§ 2. PERMUTATIONS.

1. The permutations of a, b, c, d are:

1	2	3	4	1	2	3	4
a	*ab*	*abc*	*abcd*	*b*	*ba*	*bac*	*bacd*
		abd	*abdc*			*bad*	*badc*
	ac	*acb*	*acbd*		*bc*	*bca*	*bcad*
		acd	*acdb*			*bcd*	*bcda*
	ad	*adb*	*adbc*		*bd*	*bda*	*bdac*
		adc	*adcb*			*bdc*	*bdca*
c	*ca*	*cab*	*cabd*	*d*	*da*	*dab*	*dabc*
		cad	*cadb*			*dac*	*dacb*
	cb	*cba*	*cbad*		*db*	*dba*	*dbac*
		cbd	*cbda*			*dbc*	*dbca*
	cd	*cda*	*cdab*		*dc*	*dca*	*dcab*
		cdb	*cdba*			*dcb*	*dcba*

The permutations two at a time are formed from those one at a time, by annexing to each of the latter each remaining letter in turn; those three at a time from those two at a time in like manner; and so on. Evidently the permutations thus formed are all different.

Of four things, only four permutations one at a time can be formed. And since the permutations two at a time are formed from those one at a time, none of the former are omitted. For a similar reason, none of those three and four at a time are omitted. Therefore the above representation includes all permutations of the four letters, one, two, three, and four at a time.

2. The number of permutations of n things taken r at a time is denoted by the symbol ${}_nP_r$.

Then from the enumeration of the preceding article, we have

$${}_4P_1 = 4,\ {}_4P_2 = 12,\ {}_4P_3 = 24,\ {}_4P_4 = 24.$$

3. When the number of things is large, the preceding method of enumeration becomes laborious.

The following example illustrates a method of deriving a general formula for ${}_nP_r$.

We have $${}_4P_1 = 4.$$

Each permutation one at a time gives as many permutations two at a time as there are things remaining to annex to it in turn, in this case three.

Therefore $${}_4P_2 = 3\ {}_4P_1 = 4 \times 3.$$

Each permutation two at a time gives as many permutations three at a time as there are things remaining to annex to it in turn, in this case two.

Therefore $${}_4P_3 = 2\ {}_4P_2 = 4 \times 3 \times 2.$$

In like manner, ${}_4P_4 = {}_4P_3 = 4 \times 3 \times 2 \times 1.$

In general, ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1),$

when the n things are all different.

Evidently $${}_nP_1 = n. \tag{1}$$

From each permutation of n things one at a time we obtain, by annexing to it each of the $n-1$ remaining things in turn, $n-1$ permutations two at a time.

Therefore $${}_nP_2 = (n-1)\,{}_nP_1 = n(n-1). \tag{2}$$

Again, from each permutation of n things two at a time we obtain, by annexing to it each of the $n-2$ remaining things in turn, $n-2$ permutations three at a time.

Therefore $${}_nP_3 = (n-2)\,{}_nP_2 = n(n-1)(n-2). \tag{3}$$

In like manner,

$${}_nP_4 = (n-3)\,{}_nP_3 = n(n-1)(n-2)(n-3). \tag{4}$$

The method is evidently general. The number subtracted from n in the last factor in (1)–(4) is *one less than the number of things taken at a time*. Therefore,

$${}_nP_r = n(n-1)(n-2)\cdots[n-(r-1)] = n(n-1)(n-2)\cdots(n-r+1).$$

4. Observe that the number of factors in the formula for ${}_nP_r$ is equal to the number of things taken at a time.

E.g., $${}_8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720.$$

5. If all the things are taken at a time, *i.e.*, if $r = n$, we have

$${}_nP_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots 3 \times 2 \times 1.$$

E.g., $${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

6. The continued product

$$n(n-1)(n-2)\cdots 3 \times 2 \times 1$$

is called **Factorial-*n*,** and is denoted by the symbol $\underline{|n}$ or $n!$

Therefore the formula of the preceding article may be written

$${}_nP_n = \underline{|n}.$$

E.g., $${}_7P_7 = \underline{|7}, \text{ or } 7!, = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$$

7. In many applications the things considered are not all different. We will now derive a formula for the number of permutations of n things, *taken all at a time*, when some of them are alike.

Let p of the n things be alike, and suppose the permutations n at a time to be formed. In any one of these permutations, let the p like things be replaced by p unlike things, different from all the rest. Then by changing the order of these p new things only, we can form $\underline{|p}$ permutations from the one permutation. In like manner, $\underline{|p}$ permutations can be formed from each of the given permutations. Therefore

$${}_nP_n \text{ (all different)} = \underline{|p}\ {}_nP_n \text{ } (p \text{ alike}),$$

or

$${}_nP_n \text{ } (p \text{ alike}) = \frac{{}_nP_n}{\underline{|p}} = \frac{\underline{|n}}{\underline{|p}}.$$

In like manner, it can be proved that

$${}_nP_n \text{ } (p \text{ alike, } q \text{ alike, } \cdots) = \frac{\underline{|n}}{\underline{|p} \times \underline{|q} \times \cdots}$$

E.g., $${}_8P_8 \text{ (3 alike)} = \frac{\underline{|8}}{\underline{|3}} = 6720.$$

EXERCISES L

Find the values of

1. ${}_{18}P_4$. **2.** ${}_{15}P_3$. **3.** ${}_{10}P_{10}$. **4.** ${}_{12}P_9$. **5.** ${}_{20}P_5$.

6. ${}_{n+1}P_3$. **7.** ${}_{2n+1}P_5$. **8.** ${}_{n+1}P_{n-1}$. **9.** ${}_{n+k}P_k$. **10.** ${}_{m+n}P_{m-n}$.

Find the value of n,

11. When ${}_nP_4 = 3\,{}_nP_3$. **12.** ${}_nP_6 = 20\,{}_nP_4$. **13.** ${}_{n+2}P_4 = 15\,{}_nP_3$.

14. When ${}_{n+1}P_4=30\,{}_{n-1}P_2$. **15.** ${}_{n+4}P_3=8\,{}_{n+3}P_2$. **16.** ${}_{2n+1}P_4=140\,{}_nP_3$.

Find the value of k,

17. When ${}_{10}P_{k+6}=3\,{}_{10}P_{k+5}$. **18.** ${}_7P_{k+1}=12\,{}_7P_{k-1}$. **19.** ${}_{12}P_k=20\,{}_{12}P_{k-2}$.

20. How many numbers of 4 figures can be formed with 1, 2, 3, 4, 5, 6, 7?

21. How many numbers of 4 figures can be formed with 0, 1, 2, 3, 4, 5, 6, 7?

22. How many even numbers of 4 figures can be formed with 4, 5, 3, 2?

23. In how many ways can 6 pupils be seated in 10 seats?

24. In how many ways can 4 tickets be placed in 6 different boxes, so that no box shall contain more than 1 ticket?

25. How many numbers of 5 figures can be formed with 1, 2, 3, 4, 5, 6, 7, 8, 9, if the figure 7 be in the middle of each number?

26. If the permutations of 1, 2, 3, 4, taken all at a time, be arranged in a column, how many times is each figure found in each column?

27. How many permutations can be formed with the letters in the word *Philippine?*

28. How many permutations can be formed with the letters in the word *Iloilo?*

29. In how many ways can 7 men be seated at a round table?

30. In how many ways can a bracelet be made by stringing together 7 pearls of different shades?

§ 3. COMBINATIONS.

1. The formula for the number of combinations of n things, r at a time, which is denoted by ${}_nC_r$, is most readily obtained by deriving a relation between ${}_nP_r$ and ${}_nC_r$. The method will be illustrated by a particular example.

The combinations of the four letters a, b, c, d, taken three at a time, evidently are: abc, abd, acd, bcd. From the combination abc we obtain, by changing the order of the letters in all possible ways, $\underline{|3}$ permutations. In like manner, each of the combinations gives $\underline{|3}$ permutations.

Therefore

$$_4P_3 = \underline{|3}\, _4C_3, \quad \text{or} \quad _4C_3 = \frac{_4P_3}{\underline{|3}} = \frac{4 \times 3 \times 2}{\underline{|3}}.$$

In general,

$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} = \binom{n}{r},$$

wherein the n things are all different.

For from each combination that contains r things can be formed $\underline{|r}$ permutations, by changing the order of the things in all possible ways. Therefore

$$_nP_r = \underline{|r}\, _nC_r, \quad \text{or} \quad _nC_r = \frac{_nP_r}{\underline{|r}} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}.$$

E.g.,

$$_8C_3 = \frac{8 \times 7 \times 6}{\underline{|3}} = 56.$$

2. The formula for $_nC_r$ can be put in a more convenient form for purposes of theory.

We have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1) \times (n-r)(n-r-1)\cdots 3 \times 2 \times 1}{\underline{|r} \times (n-r)(n-r-1)\cdots 3 \times 2 \times 1}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

3. We have

$$_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}},$$

and

$$_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}},$$

therefore,

$$_nC_r = {_nC_{n-r}}.$$

That is, *the number of combinations of n dissimilar things r at a time is equal to the number of combinations of the n things n − r at a time.*

This relation is also evident from the definition of a combination. For, every time that r things are taken from the n things to form a combination, there is left a combination of $n-r$ things.

E.g.,
$$ {}_{100}C_{98} = {}_{100}C_2 = \frac{100 \times 99}{1 \times 2} = 4950.$$

This relation is thus useful in computing the number of combinations when the number of things taken at a time is large.

EXERCISES II.

Find the values of

1. ${}_{11}C_5$. 2. ${}_{15}C_7$. 3. ${}_{25}C_{20}$. 4. ${}_{98}C_{95}$. 5. ${}_nC_{n-5}$.

Find the value of n,

6. When $2\,{}_nC_5 = 9\,{}_{n-2}C_5$.
7. When $3\,{}_nC_3 = 10\,{}_{n-2}C_2$.
8. When $4\,{}_{n+1}C_4 = 15\,{}_{n-1}C_3$.
9. When ${}_{n+1}P_4 = 112\,{}_{n-1}C_2$.
10. When ${}_{n+1}P_4 = 84\,{}_{n-1}C_3$.
11. When ${}_nP_2 = 24\,{}_nC_{n-1}$.

Find the value of k, when

12. ${}_8P_k = 24\,{}_8C_k$. 13. ${}_6P_{k+1} = 48\,{}_6C_k$. 14. ${}_{10}P_k = 144\,{}_{10}C_{k-1}$.

15. Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

16. In how many ways can a committee of 4 men be appointed from 25 men?

17. In how many ways can 3 books be selected from 15 books?

18. In a plane are 20 points, no 3 of which are in the same straight line. How many triangles can be formed with 3 of the points as vertices? How many quadrilaterals, with 4 of the points as vertices? How many hexagons, with 6 of the points as vertices?

19. How many triangles are formed by 7 straight lines in a plane, if no 2 are parallel and no 3 intersect in a common point? How many by 10 lines? How many by n lines?

§ 4. TWO IMPORTANT PRINCIPLES.

1. The following example illustrates an important principle.

Pr. Between two cities A and B there are five railroad lines. In how many ways can a man go from A to B and return by a different road?

He can go to B in either of five ways.

With each of these five ways he has a choice of four ways of returning. Hence he can make the round trip in $5 \times 4, = 20,$ ways.

Evidently, if he were not required to return by a different road, he could make the trip in $5 \times 5, = 25,$ ways.

The general principle is:

If one thing can be done in ***a*** *ways, and another thing can be done in* ***b*** *ways, and the doing of the first thing does not interfere with the doing of the second, the two things can be done in* ***ab*** *ways.*

The truth of the principle is evident.

2. The following relation will be useful in subsequent work:

$$ {}_{m+n}C_r = {}_mC_r + {}_mC_{r-1}\,{}_nC_1 + {}_mC_{r-2}\,{}_nC_2 + \cdots + {}_mC_2\,{}_nC_{r-2} + {}_mC_1\,{}_nC_{r-1} + {}_nC_r, \qquad (1)$$

in which $m >$ or $= r$, $n >$ or $= r$.

The number of combinations of the $m + n$ things r at a time is evidently the sum of:

The number of combinations of m things taken r at a time, or ${}_mC_r$.

The number of combinations of m things taken $r - 1$ at a time, multiplied by the number of combinations of n things taken one at a time, or ${}_mC_{r-1}\,{}_nC_1$.

And so on.

This relation may be written

$$\binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}. \qquad (2)$$

3. The relation of the preceding article requires m, n, and r to be integers. Evidently, however, the second member of (2) could be made identical with the first member by ordinary reduction. We, therefore, conclude that this relation holds for all rational values of m and n, provided r is a positive integer.

§ 5. PROBLEMS.

1. Pr. **1.** In how many ways can a committee of 3 Republicans and 4 Democrats be appointed from 18 Republicans and 12 Democrats?

The 3 Republicans can be chosen in ${}_{18}C_3, = 816,$ ways, and the 4 Democrats in ${}_{12}C_4, = 495,$ ways. Since any 3 Republicans can be associated with any 4 Democrats to form the committee the required number of ways is $816 \times 495, = 403{,}920$.

Pr. 2. A box contains 20 balls numbered 1 to 20. In how many ways can 7 balls be selected, if 1 be included, and 2, 3 be excluded?

We first set aside 1 to be included, and 2, 3 to be excluded. and from the remaining 17 balls select 6 balls. Then 1 may be combined with each of the latter in one way, giving a combination of 7 balls. Therefore the problem is equivalent to determining the number of combinations of 17 things, 6 at a time.

$$\text{Hence} \quad {}_{17}C_6 = \frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, = 12376,$$

is the required number of ways.

EXERCISES III.

1. A man has 3 coats, 4 vests, and 5 pairs of trousers. In how many ways can he dress?

2. In how many ways can 4 white balls, 3 black balls, and 2 red balls be selected from 8 white balls, 7 black balls, and 5 red balls?

3. In how many ways can permutations be formed, with 10 consonants and 4 vowels, each one to contain 5 consonants and 2 vowels?

4. In how many ways can 4 hearts, 3 diamonds, 2 clubs, and 1 spade be drawn from a pack containing 13 cards of each kind?

5. In how many ways can 7 pears, 5 apples, and 4 oranges be given to 16 children, each child to receive a piece of fruit?

6. How many numbers of 7 figures can be formed with 1, 2, 3, 4, 5, 6, 7, if the figures 4, 5, 6 be kept together?

7. In a company are 10 men, 12 women, and 15 children. In how many ways can a party be formed, consisting of 4 men, 6 women, and 10 children?

8. How many permutations of 10 letters can be formed from 5 consonants and 5 vowels, if no two consonants be adjacent?

9. How many permutations of 9 letters can be formed from 5 consonants and 4 vowels, if each vowel be placed between two consonants?

10. In how many ways can a committee of 5 men be appointed from 20 men, if there be no restriction in the choice? In how many ways, if a particular man be always chosen? In how many ways, if a particular man be always excluded?

11. In a school are 96 pupils. In how many ways can a teacher divide them into sections of 12?

12. In how many ways can 4 ladies and 4 gentlemen be seated at a square table, so that a gentleman and a lady shall be seated at each side?

13. How many throws can be made with 2 dice, if such throws as 1, 2 and 2, 1 be regarded as the same? How many with 3 dice?

14. In how many ways can the sum 10 be thrown with 2 dice? With 3 dice?

15. In how many ways can 52 cards be divided into 4 sets of 13 cards each?

16. A box contains 15 balls, numbered 1 to 15. In how many ways can 5 balls be selected, if 1, 2, 3 be included? In how many ways, if 1, 2 be included, and 3 excluded? In how many ways, if any two of the numbers 1, 2, 3 be included, the other excluded?

17. In how many ways can 10 different coins be arranged in a row, if the faces of each coin are distinct? In how many ways can they be arranged in a circle?

18. In how many ways can a number of 6 figures be formed with 1, 1, 1, 2, 2, 3, the first and last figure of each number to be an even digit?

19. A man can go to his office in 3 ways. In how many ways can he arrange to go to his office for 6 days?

20. In how many ways can 7 gentlemen and 10 ladies arrange a game of lawn tennis, each side to consist of 1 lady and 1 gentleman?

CHAPTER XXX.

VARIABLES AND LIMITS.

§ 1. VARIABLES.

1. A **Variable** is a number that may have a series of different values in the same investigation or problem.

A **Constant** is a number that has a fixed value in an investigation or problem.

Thus, if d is the number of feet a body has fallen from rest in s seconds, it has been shown by experiment that

$$d = 16 s^2.$$

As the body falls, the distance d and the time s are variables, and 16 is a constant.

Again, time measured from a past date is a variable, while time measured between two fixed dates is a constant.

2. The constants in a mathematical investigation are, as a rule, general numbers, and are represented by the first letters of the alphabet, a, b, c, etc.; variables are usually represented by the last letters, x, y, z, etc.

3. A variable which has a definite value, or set of values, corresponding to a value of a second variable, is called a **Function** of the latter.

Thus, $16 x^2$, $\pm \sqrt{(a^2 - x^2)}$, etc., are functions of x; corresponding to any value of x, the first function has one value, the second has two values.

Again, the area of a circle is a function of its radius; the distance a train runs is a function of the time and speed.

4. Much simplicity is introduced into mathematical investigations by employing special symbols for functions.

The symbol $f(x)$, read *function of x*, is very commonly used to denote a function of x.

Thus, $f(x)$ may denote x^3+1 in one investigation, ax^2+bx+c in another.

5. The result of substituting a particular value for the variable in a given expression may be indicated by substituting the same value for the variable in the functional symbol.

Thus, if $f(x)=x^2+1$, then $f(a)=a^2+1$, $f(2)=2^2+1=5$, $f(0)=0+1=1$.

EXERCISES I.

1. Given $f(x)=5x^2-3x+2$; find $f(4)$; $f(3)$, $f(0)$, $f(-4)$, $f(x^2)$.
2. Given $f(x)=a^x$; find $f(0)$, $f(4)$, $f(-5)$, $f(x^2)$, $f(a)$.
3. Given $f(m)=1+mx+\frac{m(m-1)}{\underline{|2}}x^2+\cdots$; find $f(5)$, $f(\frac{2}{3})$, $f(-8)$, $f(0)$.

§ 2. LIMITS.

1. When the difference between a variable and a constant may become and remain less than any assigned positive number, however small, the constant is called the **Limit** of the variable.

Let the point P move from A towards B (Fig. 4) in the following way: First to P_1, one-half of the distance from A to B; next from P_1 to P_2, one-half of the distance from P_1 to B; then from P_2 to P_3, one-half of the distance from P_2 to B; and so on.

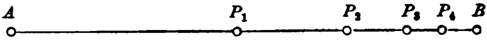

FIG. 4.

Evidently, as P thus moves from A to B, its distance from A becomes more and more nearly equal to AB, and the difference between AP and AB can be made less than any assigned distance, however small, by continuing indefinitely the motion of P. Therefore AB is the limit of AP.

If we call the distance from A to B unity, we have

$$AP_1 = \tfrac{1}{2},\ P_1P_2 = \tfrac{1}{4},\ P_2P_3 = \tfrac{1}{8},\ P_3P_4 = \tfrac{1}{16},\ \cdots.$$

Hence,

$$AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$$

But, by Ch. XXVII., § 3, Art. 8, the sum of the series on the right approaches 1 as a limit. That is,

$$\text{limit of } (AP_1 + P_1P_2 + {}_2PP_3 + P_3P_4 + \cdots) = AB.$$

Again, $1 + \frac{1}{n}$ becomes more and more nearly equal to 1, as n increases indefinitely, and $\left(1 + \frac{1}{n}\right) - 1, = \frac{1}{n}$, will become and remain less than any assigned positive number, however small.

2. It follows from the definition of a limit that the variable may be always greater, or always less, or sometimes greater and sometimes less than its limit.

Thus, by Ch. XXVII., § 3, Art. 8, we have

$$\text{limit } (1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8} - \cdots) = 0, \quad (1)$$

$$\text{limit } (1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots) = 2, \quad (2)$$

$$\text{limit } (1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots) = \tfrac{2}{3}. \quad (3)$$

And in (1), $S_1 = 1,\ S_2 = \frac{1}{2},\ S_3 = \frac{1}{4},\ S_4 = \frac{1}{8},\ \cdots;$ (4)

in (2), $S_1 = 1,\ S_2 = \frac{3}{2},\ S_3 = \frac{7}{4},\ S_4 = \frac{15}{8},\ \cdots;$ (5)

in (3), $S_1 = 1,\ S_2 = \frac{1}{2},\ S_3 = \frac{3}{4},\ S_4 = \frac{5}{8},\ \cdots.$ (6)

3. The symbol, $\doteq$, read *approaches as a limit,* or simply *approaches,* is placed between a variable and its limit.

The word limit may be abbreviated to lim.

Thus, $\lim\limits_{x \doteq 1}(1 - x) = 0$, read *the limit of* $1 - x$, *as* x *approaches* 1, *is* 0.

4. The difference between a variable and its limit is evidently a variable whose limit is 0. That is,

if lim $x = a$, then lim $(x - a) = 0$.

5. If the limit of a variable be 0, the limit of the product of the variable and any finite number is 0. That is,

$$\text{if } \lim x = 0, \text{ and } a \text{ be any finite number, } \lim ax = 0.$$

Let k be any number, however small. Then x can be made less numerically than $\frac{k}{a}$, and, therefore, ax less than k. Hence, $\lim ax = 0$.

Fundamental Principles of Limits.

6. (i.) *If two variables be always equal, and one of them approach a limit, the other approaches the same limit.* That is,

$$\text{if } x = y, \text{ and } x \doteq a, \text{ then } y \doteq a.$$

(ii.) *If two variables be always equal as they approach their limits, their limits are equal.* That is,

$$\text{if } \lim x = a,\ \lim y = b, \text{ and } x = y, \text{ then } a = b.$$

(iii.) *The limit of the algebraical sum of a finite number of variables is the sum of their limits.* That is,

$$\text{if } \lim x = a,\ \lim y = b,\ \cdots, \text{ then } \lim(x + y + \cdots) = a + b + \cdots.$$

(iv.) *The limit of the product of a finite number of variables is the product of their limits, if none of the limits be* ∞. That is,

$$\text{if } \lim x = a,\ \lim y = b,\ \cdots, \text{ then } \lim(xy\cdots) = ab\cdots.$$

(v.) *The limit of the quotient of two variables is the quotient of their limits, if the limit of* $y \neq 0$. That is,

$$\text{if } \lim x = a,\ \lim y = b, \text{ then } \lim\left(\frac{x}{y}\right) = \frac{a}{b}.$$

The proofs follow:

(i.) We have $x = a + x'$, wherein, by Art. 4, x' is a variable whose limit is 0. Then, since $y = x$ always, we have $y = a + x'$. Hence $\lim y = a$.

(ii.) This principle follows directly from (i.).

(iii.) We have $x = a + x'$, $y = b + y$, $\cdots$, wherein, by Art. 4, x', y' $\cdots$ are variables whose limits are 0, and which can therefore be made numerically less than any assigned number, however small.

Then $x + y + \cdots = (a + b + \cdots) + (x' + y' + \cdots)$.

Let k be any assigned number, however small. Then each of the variables x', y', $\cdots$ can be made less than $\frac{k}{n}$, wherein n is the number of variables. Therefore, $x' + y' + \cdots$ can be made less than k. Consequently $\lim (x + y \cdots) = a + b + \cdots$.

(iv.) We have $x = a + x'$, $y = b + y'$, $\cdots$, wherein x', y', $\cdots$ are variables whose limits are 0, and a, b, $\cdots$ are finite.

Then $xy \cdots = ab + bx' + ay' +$ a finite number of terms each of which has one or more of the factors x', y', $\cdots$.

Therefore, by (iii.),

$$\begin{aligned}\lim (xy \cdots) &= \lim ab + \lim bx' + \lim ay' + \cdots \\ &= ab, \text{ since } \lim bx' = 0, \cdots, \text{ by Art. 5.}\end{aligned}$$

(v.) Let $\frac{x}{y} = q$, or $x = yq$.

Then, by (iv.), $\lim x = \lim y \lim q$.

Therefore $\qquad \lim q = \frac{\lim x}{\lim y}$, or $\lim \frac{x}{y} = \frac{\lim x}{\lim y}$.

Infinitesimals and Infinites.

7. A variable which may become and remain numerically less than any assigned positive number, however small, is called an **Infinitesimal.**

A variable which may become and remain numerically greater than any assigned positive number, however great, is called an **Infinite Number**, or simply an **Infinite**.

It is important to keep in mind that both infinitesimals and infinites are variable.

8. The conclusions reached in Ch. III., § 4, Arts. 14 and 19, may be restated thus:

$$\frac{N}{x} \doteq \infty, \text{ as } x \doteq 0; \text{ and } \frac{N}{x} \doteq 0, \text{ as } x \doteq \infty.$$

Indeterminate Fractions.

9. The fraction $\frac{x^2 - 9}{x - 3}$ becomes $\frac{0}{0}$ when $x = 3$, and has no definite value. But so long as $x \neq 3$, however little it may differ from 3, we may perform the indicated division. We therefore have

$$\frac{x^2-9}{x-3}=x+3, \text{ when } x \neq 3.$$

Now, since the limit of the fraction depends upon values of x which differ from 3, however little, we have

$$\lim_{x \doteq 3}\frac{x^2-9}{x-3}=\lim_{x \doteq 3}(x+3)=6.$$

Although the given fraction is indeterminate, it is clearly desirable that it shall have a definite value. *We therefore assign to* $\frac{x^2-9}{x-3}$ *the value* 6, *when* $x=3$.

That is, we *define* an indeterminate fraction to be the limit of the fraction as the variable approaches that value which renders it indeterminate.

EXERCISES II.

Find the limiting values of the following fractions

1. $\dfrac{x^2-6x+5}{x^2-8x+15}$, when $x \doteq 5$.

2. $\dfrac{x^2-3x+2}{x^2+x-6}$, when $x \doteq 2$.

3. $\dfrac{3a^2-ab-2b^2}{9a^2+12ab+4b^2}$, when $a \doteq -\frac{2}{3}b$.

4. $\dfrac{9x^2-30xy+25y^2}{3x^2-2xy-5y^2}$, when $x \doteq \frac{5}{3}y$.

5. $\dfrac{x^3+2x^2-x-2}{x^2+x-2}$, when $x \doteq 1$.

6. $\dfrac{x^3-2x+2}{x^2-6x+5}$, when $x \doteq 1$.

7. $\dfrac{x^2-6x+5}{x^3-2x+2}$, when $x \doteq 1$.

8. $\dfrac{a^{2x}-1}{a^x-1}$, when $x \doteq 0$.

9. $\dfrac{2ac-a^2+b^2-c^2}{2ab-a^2-b^2+c^2}$, when $c \doteq a-b$.

10. $\dfrac{x^2+2xy+y^2-z^2}{x^2+2xz-y^2+z^2}$, when $x \doteq -(y+z)$.

CHAPTER XXXI.

CONVERGENT AND DIVERGENT SERIES.

1. In this chapter we shall briefly discuss the nature of infinite series.

It follows from Ch. XXVII., § 3, Art. 8, that the sum of n terms of the decreasing geometrical progression

$$a + ar + ar^2 + \cdots + ar^n + \cdots,$$

wherein r is numerically less than 1, approaches $\dfrac{a}{1-r}$ as a limit, as n increases indefinitely.

Let $$S = 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \cdots.$$

We then have $$S = \frac{1}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3},$$

and $S_1 = 1,\ S_2 = \frac{1}{2},\ S_3 = \frac{3}{4},\ S_4 = \frac{5}{8},\ S_5 = \frac{11}{16},\ S_6 = \frac{21}{32}, \cdots.$

Evidently these sums approach $\frac{2}{3}$ more and more nearly, as more and more terms are included.

This infinite series may therefore be regarded as having the finite sum $\frac{2}{3}$.

But the sum of the series

$$1 + 2 + 4 + 8 + \cdots$$

increases beyond any finite number, as the number of terms increases indefinitely.

2. The examples of the preceding article illustrate the following definitions:

An infinite series is said to be **Convergent**, when the sum of the first n terms approaches a definite finite limit, as n increases indefinitely.

An infinite series is said to be **Divergent** when the sum of the first n terms increases numerically beyond any assigned number, however great, as n increases indefinitely.

3. Infinite series arise in connection with many mathematical operations. Thus, for example, if the division of 1 by $1-x$ be continued indefinitely, we obtain as a quotient the infinite series

$$1+x+x^2+x^3+\cdots+x^n+\cdots.$$

This series is, by Art. 1, convergent when x is numerically less than 1. Evidently, when $x=1$, the series is divergent. When $x=-1$, we have $1-1+1-1+\cdots$. The sum of n terms of this series is $+1$ or 0, according as n is odd or even. The series is said to *oscillate* and is neither convergent nor divergent. When x is numerically greater than 1, we have, by Ch. XXVII., § 3, Art. 4,

$$S_n=\frac{1-x^n}{1-x}.$$

By taking n sufficiently great this expression can be made to exceed numerically any number, however great. Therefore the series is then divergent.

It is obvious that we can regard the series as equivalent to $\frac{1}{1-x}$ only when it is convergent, that is, when x is numerically less than 1. Thus it becomes important to decide whether a series which arises in connection with a mathematical operation is convergent or divergent.

4. In the theory which follows we shall let S stand for the limit of the sum of n terms of the series

$$u_1+u_2+\cdots+u_n+\cdots,$$

as n increases indefinitely.

Also let $$S_n=u_1+u_2+\cdots+u_n,$$
and $${}_mR_n=u_{n+1}+u_{n+2}+\cdots+u_{n+m};$$
that is, ${}_mR_n$ denotes the sum of m terms after the first n terms,

Then $$S_n+{}_mR_n=S_{n+m}.$$

5. *The series $u_1 + u_2 + \cdots + u_n + \cdots$ is convergent if S_n remain finite, and ${}_mR_n$ approach 0 for all values of m, as n increases indefinitely; and, conversely, if the series be convergent, these two conditions are satisfied.*

For, by the first condition, the limit of S_n is finite. By the second condition,

$$\lim (S_{n+m} - S_n), = \lim {}_mR_n, = 0.$$

Therefore $\lim S_{n+m} = \lim S_n$. Hence, since S_n cannot have one finite limit for one value of n, and a different finite limit for another value of n, the limit of S_n is a *definite* finite number.

Also, if the series is convergent, S_n must be finite by definition; and, since $\lim S_{n+m} = \lim S_n$, we have

$$\lim (S_{n+m} - S_n), = \lim {}_mR_n, = 0.$$

6. *If a series be convergent when its terms are all positive, it is convergent when some or all of them are made negative.*

For, if S_n is finite, and ${}_mR_n$ approaches 0, as n increases indefinitely, with greater reason S_n is finite, and ${}_mR_n$ approaches 0, when some or all of the terms are made negative.

7. A series which is convergent when all its negative terms are made positive is said to be **Absolutely Convergent.**

8. *If the sum of n terms of an infinite series of positive terms remain finite, as n increases indefinitely, the series is convergent.*

For, since the sum continually increases, but remains finite, it must approach some finite number as a limit.

9. *An infinite series is convergent if, after some particular term, the ratio of each term to the preceding be numerically less than some fixed positive number which is itself less than unity.*

Let the series be $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$, and let the ratio of each term after the kth be numerically less than r, which is itself less than 1.

First, let the terms be all positive.

Then, from $$\frac{u_{k+1}}{u_k} < r, \ \frac{u_{k+2}}{u_{k+1}} < r, \ \frac{u_{k+3}}{u_{k+2}} < r, \ \cdots,$$

we obtain $u_{k+1} < ru_k$, $u_{k+2} < ru_{k+1} < r^2u_k$, $u_{k+3} < ru_{k+2} < r^3u_k$, $\cdots$.

Hence

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots < u_k(r + r^2 + r^3 + \cdots), \doteq u_k \cdot \frac{r}{1-r}, \text{ since } r < 1.$$

Hence, since the sum of the finite number of terms $u_1 + u_2 + \cdots + u_k$ is finite, the entire sum must be finite. Therefore, by Art. 8, the given series is convergent; and, by Art. 6, it is absolutely convergent.

10. *If an infinite series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ be convergent for values of x greater than 0, the sum of the series approaches a_0, as x approaches 0.*

Let
$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = a_0 + xS_1,$$
wherein
$$S_1 = a_1 + a_2x + a_3x^2 + \cdots.$$

Since, by the preceding article, the series S_1 is convergent for all values of x for which the given series is convergent, its sum approaches a finite limit as $x \doteq 0$, and consequently $xS_1 \doteq 0$, when $x \doteq 0$. Hence

$$a_0 + a_1x + a_2x^2 + \cdots, = a_0 + xS_1 \doteq a_0 \text{ when } x \doteq 0.$$

11. *If two integral series, arranged to ascending powers of x, be equal for all values of x which make them both convergent, the coefficients of like powers of x are equal.*

Let
$$a_0 + a_1x + a_2x^2 + \cdots = b_0 + b_1x + b_2x^2 + \cdots$$
for all values of x which make the two series convergent.

Then the sums of the two series approach equal limits when $x \doteq 0$. But, by the preceding article, the sum of the one series approaches a_0, that of the other b_0; consequently $a_0 = b_0$,

and
$$a_1x + a_2x^2 + \cdots = b_1x + b_2x^2 + \cdots.$$

Since these two series are convergent for all values of x for which the original series are convergent, they are equal for values of x other than zero, and the last equation may be divided by x.

Hence
$$a_1 + a_2x + a_3x^2 + \cdots = b_1 + b_2x + b_3x^2 + \cdots;$$
and, as before,
$$a_1 = b_1,$$
and
$$a_2x + a_3x^2 + \cdots = b_2x + b_3x^2 + \cdots.$$

In like manner, we can prove $a_2 = b_2$, $a_3 = b_3$, etc.

12. Evidently the principle of the preceding article holds with greater reason if either or both of the series be finite, *i.e.* have a limited number of terms. There is, in this case, no question of convergence of the finite series. The series must be equal for all values of x, if they be both finite; or, if one be infinite, for all values of x which make that series convergent.

13. The following application of the principle of Art. 9 will be useful in the next chapter.

The series

$$1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots$$

is absolutely convergent when $x<1$ *numerically.*

For
$$\frac{u_{k+1}}{u_k}=\frac{\binom{n}{k}x^k}{\binom{n}{k-1}x^{k-1}}=\frac{n-k+1}{k}x=\left(\frac{n+1}{k}-1\right)x.$$

Hence for all values of $k>n+1$, this ratio will be numerically less than the absolute value of x. Consequently, when $x<1$ numerically, the series is absolutely convergent by Art. 9.

EXERCISES.

Examine the following series with respect to their convergency or divergency:

1. $\frac{3}{4}+\frac{3\cdot 4}{4\cdot 6}+\frac{3\cdot 4\cdot 5}{4\cdot 6\cdot 8}+\cdots+\frac{3\cdot 4\cdots(n+2)}{4\cdot 6\cdots(2n+2)}+\cdots.$

2. $\frac{3}{4}+\frac{3\cdot 5}{4\cdot 7}+\frac{3\cdot 5\cdot 7}{4\cdot 7\cdot 10}+\cdots+\frac{3\cdot 5\cdots(2n+1)}{4\cdot 7\cdots(3n+1)}+\cdots.$

3. $\frac{1}{3}+\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\ 9}+\cdots+\frac{1\cdot 3\cdots(2n-1)}{3\cdot 6\cdots 3n}+\cdots.$

4. $1+2x+3x^2+4x^3+\cdots.$

5. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots.$

6. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\frac{x^4}{4\cdot 5}+\cdots.$

7. $x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\cdots.$

8. $\frac{x}{\sqrt{(1\cdot 2)}}+\frac{x^2}{\sqrt{(2\cdot 3)}}+\frac{x^3}{\sqrt{(3\cdot 4)}}+\frac{x^4}{\sqrt{(4\cdot 5)}}+\cdots.$

CHAPTER XXXII.

THE BINOMIAL THEOREM.

§ 1. THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS.

1. In Ch. XXVIII., it was proved by induction that, when n is a positive integer,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

We will here give a briefer proof, based upon the theory of combinations.

Consider the following continued product of n factors:

$$n \text{ factors} \left\{ \begin{array}{c} a+b \\ a+b \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ a+b \end{array} \right.$$

The first term of the product is formed by taking an a from each factor, giving a^n. The second term is formed by taking an a from $n-1$ factors and a b from the remaining factor, giving $a^{n-1}b$. But such a term can be formed in as many ways as one b can be taken from n b's, *i.e.*, in ${}_nC_1$ ways. Therefore the product so far is $a^n + {}_nC_1a^{n-1}b$.

A third term is formed by taking an a from $n-2$ factors and a b from the remaining two factors, giving $a^{n-2}b^2$. But such a term can be formed in as many ways as two b's can be taken from n b's, *i.e.*, in ${}_nC_2$ ways. Consequently, the product to this point is $a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2$.

In general, an a can be taken from each of $n-k+1$ factors and a b from each of the remaining $k-1$ factors, giving $a^{n-k+1}b^{k-1}$. But such a term can evidently be formed in ${}_nC_{k-1}$ ways.

We thus obtain

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_{k-1}a^{n-k+1}b^{k-1} + \cdots,$$

But $${}_nC_1=\binom{n}{1},\ {}_nC_2=\binom{n}{2},\ {}_nC_3=\binom{n}{3},\ {}_nC_{k-1}=\binom{n}{k-1}.$$

Therefore, $$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\binom{n}{3}a^{n-3}b^3$$
$$+\cdots+\binom{n}{k-1}a^{n-k+1}b^{k-1}+\cdots.$$

2. The kth term, counting from the beginning of the expansion, contains b^{k-1}, and is ${}_nC_{k-1}a^{n-k+1}b^{k-1}$. The kth term, counting from the end, contains a^{k-1}, and therefore b^{n-k+1}, and is ${}_nC_{n-k+1}a^{k-1}b^{n-k+1}$. But, by Ch. XXIX., § 3, Art. 3, we have ${}_nC_{k-1}={}_nC_{n-k+1}$. We therefore conclude:

In the expansion of $(a+b)^n$, *wherein* n *is a positive integer, the coefficients of terms equally distant from the beginning and end of the expansion are equal.*

§ 2. BINOMIAL THEOREM FOR ANY RATIONAL EXPONENT.

1. From Ch. XXVIII., Art. 4, we have

$$(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots, \tag{1}$$

when n is a positive integer. In this case the expansion ends with the $n+1$th term, since the coefficients of the $n+2$th and all succeeding terms contain $n-n$, or 0, as a factor. But if n be not a positive integer, the expression on the right of (1) will continue without end, since no factor of the form $n-k+1$ can reduce to 0. Therefore this series will have no meaning unless it be convergent.

2. In Ch. XXXI., Art. 13, it was proved that the series

$$1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots$$

is convergent when x lies between -1 and $+1$. It remains to be proved, therefore, that in this case the above series represents $(1+x)^n$, when n is a fraction or negative.

3. Since the reasoning will turn upon the value of n, we shall call the expression

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

a function of n, and abbreviate it by $f(n)$, *for all rational values of n.* To understand the following reasoning, the student should notice that for all positive integral values of n, $(1 + x)^n = f(n)$, as, $(1 + x)^3 = f(3)$; and that it remains to prove that $(1 + x)^n = f(n)$, when n is a fraction or negative, as, for example, that $(1 + x)^{\frac{2}{3}} = f(\frac{2}{3})$.

4. We now have

$$f(m) = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{k-1}x^{k-1} + \cdots$$

$$f(n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k-1}x^{k-1} + \cdots$$

for real values of x between -1 and $+1$.

We will tentatively assume, what will be proved in Part II., Text-Book of Algebra, that the product

$$\begin{aligned} f(m) \times f(n) = 1 &+ \left[\binom{m}{1} + \binom{n}{1}\right]x + \left[\binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2}\right]x^2 + \cdots \\ &+ \left[\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \binom{m}{k-3}\binom{n}{2} + \cdots \right. \\ &\left. + \binom{m}{2}\binom{n}{k-3} + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1}\right]x^{k-1} + \cdots \end{aligned}$$

is convergent. But

$$\binom{m}{1} + \binom{n}{1} = \binom{m+n}{1}, \quad \binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2} = \binom{m+n}{2},$$

and by Ch. XXIX., § 4, Art. 2,

$$\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1} = \binom{m+n}{k-1};$$

therefore
$$f(m) \times f(n) = f(m + n), \tag{1}$$
for all rational values of m and n.

Then $f(m) \times f(n) \times f(p) = f(m + n) \times f(p) = f(m + n + p)$.

In general,

$$f(m) \times f(n) \times f(p) \times \cdots \times f(r) = f(m + n + p + \cdots + r), \tag{2}$$

for all rational values of m, n, p, $\cdots$, r.

5. Fractional Exponents. — Let

$$m = n = p = \cdots = r = \frac{u}{v},$$

wherein u and v are positive integers. Taking v factors, we now have

$$f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times \cdots v \text{ factors} = f\left(\frac{u}{v} + \frac{u}{v} + \frac{u}{v} + \cdots v \text{ summands}\right),$$

or

$$\left[f\left(\frac{u}{v}\right)\right]^v = f\left(\frac{u}{v} \cdot v\right) = f(u).$$

Now, since u is a positive integer,

$$(1 + x)^u = f(u).$$

Therefore $\quad (1 + x)^u = \left[f\left(\frac{u}{v}\right)\right]^v$, or $(1 + x)^{\frac{u}{v}} = f\left(\frac{u}{v}\right)$.

That is, $\quad (1 + x)^{\frac{u}{v}} = 1 + \left(\begin{matrix} \frac{u}{v} \\ 1 \end{matrix}\right) x + \left(\begin{matrix} \frac{u}{v} \\ 2 \end{matrix}\right) x^2 + \cdots$.

6. Negative Exponents, Integral or Fractional. — In (1), Art. 4, let

$$m = - n.$$

We then have $\quad f(- n) \times f(n) = f(n - n) = f(0) = 1,$

since $\quad f(0) = 1 + 0 \cdot x + \cdots = 1.$

Therefore $\quad \dfrac{1}{f(n)} = f(- n).$

Since n is a positive integer or fraction, $(1 + x)^n = f(n)$, and therefore

$$\frac{1}{(1 + x)^n} = f(- n), \text{ or } (1 + x)^{-n} = f(- n).$$

That is, $\quad (1 + x)^{-n} = 1 + \binom{-n}{1} x + \binom{-n}{2} x^2 + \cdots$.

7. Expansion of $(a + b)^n$. — We have

$$(a + b)^n = \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n\left(1 + \frac{b}{a}\right)^n, \quad (1)$$

and

$$(a + b)^n = \left[b\left(1 + \frac{a}{b}\right)\right]^n = b^n\left(1 + \frac{a}{b}\right)^n. \quad (2)$$

When b is numerically less than a,

$$\left(1+\frac{b}{a}\right)^n = 1 + \binom{n}{1}\frac{b}{a} + \binom{n}{2}\frac{b^2}{a^2} + \cdots,$$

and, by (1) above,

$$(a+b)^n = a^n\left[1 + \binom{n}{1}\frac{b}{a} + \binom{n}{2}\frac{b^2}{a^2} + \cdots\right]$$

$$= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots. \qquad (3)$$

In a similar way it can be shown that, when a is numerically less than b,

$$(a+b)^n = b^n + \binom{n}{1}b^{n-1}a + \binom{n}{2}b^{n-2}a^2 + \cdots. \qquad (4)$$

Notice that when n is a fraction or negative, formula (3) or (4) must be used according as a is numerically greater or less than b.

8. Ex. Expand $\dfrac{1}{\sqrt[3]{(a-4b^2)}}$ to four terms.

If we assume $a > 4b^2$, we have, by (3), Art. 7,

$$\frac{1}{\sqrt[3]{(a-4b^2)}} = (a-4b^2)^{-\frac{1}{3}} = a^{-\frac{1}{3}} + (-\tfrac{1}{3})a^{-\frac{4}{3}}(-4b^2)$$

$$+ \frac{-\frac{1}{3}(-\frac{4}{3})}{1\cdot 2}a^{-\frac{7}{3}}(-4b^2)^2$$

$$+ \frac{-\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})}{1\cdot 2\cdot 3}a^{-\frac{10}{3}}(-4b^2)^3 + \cdots$$

$$= \frac{1}{\sqrt[3]{a}} + \frac{4b^2}{3a\sqrt[3]{a}} + \frac{32b^4}{9a^2\sqrt[3]{a}} + \frac{896b^6}{81a^3\sqrt[3]{a}} + \cdots.$$

If $a < 4b^2$, we should have used (4), Art. 7.

Any particular term can be written as in Ch. XXVIII., Art. 6.

9. Extraction of Roots of Numbers.—Ex. Find $\sqrt{17}$ to four decimal places. We have

$$\sqrt{17}=\sqrt{(16+1)}=4\left(1+\tfrac{1}{16}\right)^{\frac{1}{2}}$$

$$=4\left[1+\frac{1}{2}\times\frac{1}{16}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}\left(\frac{1}{16}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3}\left(\frac{1}{16}\right)^3+\cdots\right]$$

$$=4(1+.03125-.00048+.00001-\cdots)$$

$$=4\times 1.03078=4.12312.$$

Therefore $\sqrt{17}=4.1231$, to four decimal places.

EXERCISES.

Expand to four terms

1. $(1+a)^{\frac{1}{2}}$. 2. $(1-x)^{-1}$. 3. $(x^2+y)^{-\frac{2}{3}}$. 4. $(x-y^2)^{-4}$.

5. $(27+5x)^{\frac{2}{3}}$. 6. $(8a^3-3b)^{-\frac{1}{3}}$. 7. $(3+2x)^{\frac{3}{4}}$. 8. $(5a^2-3b^3)^{-\frac{2}{3}}$.

9. $(2x^{\frac{1}{2}}+xy^{-2})^{-3}$. 10. $[\sqrt[3]{(ab)}-\sqrt{(ab)}]^{\frac{1}{4}}$. 11. $(a^2x^{-\frac{1}{3}}-a^{-\frac{2}{3}}x^{\frac{1}{2}})^{-4}$.

12. $\dfrac{1}{\sqrt{(a^2-b^2)}}$. 13. $\dfrac{1}{\sqrt[3]{(a^3-b)}}$. 14. $\dfrac{1}{\sqrt{(2x^{-1}-34^{\frac{1}{2}})^3}}$.

Find the

15. 4th term of $(1-2x)^{\frac{1}{3}}$. 16. 6th term of $(1+a^2b^{-\frac{1}{3}})^{-3}$.

17. 5th term of $(x^{\frac{2}{3}}-x^{-1}y^2)^{-\frac{3}{4}}$. 18. 8th term of $(a^3\sqrt{b}-2b\sqrt[3]{a})^{-\frac{1}{2}}$.

19. $k-5$th term of $(1+x^{\frac{1}{3}}y^{\frac{1}{2}})^{-2}$. 20. $2k$th term of $[x^2-\sqrt{(xy)}]^{\frac{2}{3}}$.

21. Find the term in $(3x^3-x^2y)^{\frac{5}{3}}$ containing x^2.

22. Find the term in $\left(a+\dfrac{1}{2\sqrt{a}}\right)^{-\frac{1}{2}}$ containing a^{-11}.

Find to four places of decimals the values of

23. $\sqrt{5}$. 24. $\sqrt{27}$. 25. $\sqrt[3]{35}$. 26. $\sqrt[4]{700}$. 27. $\sqrt[5]{258}$.

CHAPTER XXXIII.

UNDETERMINED COEFFICIENTS.

§ 1. METHODS OF UNDETERMINED COEFFICIENTS.

1. It is frequently necessary to change an algebraical expression from one form to another. One method consists in equating the given expression to an expression of the required form, in which some or all of the coefficients are at first unknown, but can subsequently be determined.

In applying this *method of undetermined coefficients*, it should first be shown that the assumption of the required form is legitimate, that is, that the given expression can assume that form.

The method depends upon the principle of series proved in Ch. XXXI., Art. 11.

§ 2. EXPANSION OF CERTAIN FUNCTIONS INTO INFINITE SERIES.

1. It follows from the principle proved in Ch. XXXI., Art. 3, that the infinite series

$$1 + x + x^2 + x^3 + \cdots$$

approaches in value the fraction $\frac{1}{1-x}$ for all values of x between -1 and $+1$. Conversely, we may look upon the series as the expansion of the fraction for all values of x between these limits, but for no other values of x.

And, in general, an infinite series, no matter how obtained from a given function, can be regarded as the expansion of that function *only when the series is convergent.*

This fact should be kept in mind, without further emphasis, in all the expansions that we shall derive.

Rational Fractions.

2. In assuming as the expansion of a rational fraction an infinite series of ascending powers of x, we first determine with what power the series should commence. This is done by division, when both numerator and denominator are arranged to ascending powers of x. In fact, this step also determines completely the first term of the series.

Ex. 1. Expand $\dfrac{2 - x}{1 + x - x^2}$,

in a series, to ascending powers of x.

Since the first term of the expansion is evidently 2, we assume

$$\frac{2 - x}{1 + x - x^2} = 2 + Bx + Cx^2 + Dx^3 + Ex^4 \cdots,$$

wherein $B, C, D, \cdots$ are constants to be determined.

Clearing the equation of fractions, we obtain

$$2 - x = \begin{array}{r|r|r|r|l} 2 + B & x + C & x^2 + D & x^3 + E & x^4 + \cdots. \\ 2 & +B & +C & +D & \\ & -2 & -B & -C & \end{array}$$

The series on the right is infinite; that on the left may be regarded as an infinite series with zero coefficients of all powers of x higher than the first. By Ch. XXXI., Art. 11, we have

$$\begin{aligned} 2 = 2; \qquad B + 2 &= -1, \quad \text{whence } B = -3; \\ C + B - 2 &= 0, \quad \text{whence } C = 5; \\ D + C - B &= 0, \quad \text{whence } D = -8; \\ E + D - C &= 0, \quad \text{whence } E = 13; \\ \text{etc.,} & \qquad\qquad \text{etc.} \end{aligned}$$

Hence, substituting these values of $B, C, D, \cdots$ in the assumed series, we have

$$\frac{2 - x}{1 + x - x^2} = 2 - 3x + 5x^2 - 8x^3 + 13x^4 + \cdots.$$

Let the student compare this result with that obtained by division. In fact, the latter method of expanding a fraction is to be preferred when only a few terms are wanted. But the successive coefficients, after a certain stage, may be computed with great facility by the method of undetermined coefficients. A moment's inspection of the preceding work will convince the student that the coefficient D, and all which follow it, are each connected with the two immediately preceding coefficients by a definite relation. Thus,

$$D+C-B=0,\quad E+D-C=0,\quad F+E-D=0, \text{ etc.}$$

Ex. 2. Expand $$\frac{1-x}{3x^2-x^3},$$
in a series, to ascending powers of x.

The first term in the expansion is evidently $\frac{1}{3}x^{-2}$.

We therefore assume

$$\frac{1-x}{3x^2-x^3}=\tfrac{1}{3}x^{-2}+Bx^{-1}+C+Dx+Ex^2+Fx^3+\cdots.$$

Clearing of fractions, we obtain

$$1-x=1+\begin{array}{c|c|c|c} 3B & x+3C & x^2+3D & x^3+\cdots. \\ -\frac{1}{3} & -B & -C & -\cdots. \end{array}$$

By Ch. XXXI., Art. 11, we have

$$\begin{aligned} 1=1.\quad 3B-\tfrac{1}{3}&=-1, &&\text{whence } B=-\tfrac{2}{9};\\ 3C-B&=0, &&\text{whence } C=-\tfrac{2}{27};\\ 3D-C&=0, &&\text{whence } D=-\tfrac{2}{81};\\ &\text{etc.,} && \qquad\text{etc.} \end{aligned}$$

Hence $$\frac{1-x}{3x^2-x^3}=\tfrac{1}{3}x^{-2}-\tfrac{2}{9}x^{-1}-\tfrac{2}{27}-\tfrac{2}{81}x-\cdots.$$

EXERCISES I.

Expand the following fractions in series, to ascending powers of x, to four terms:

1. $\dfrac{1+x}{1-x}$.
2. $\dfrac{1+2x}{1+x+x^2}$.
3. $\dfrac{2+x-3x^2}{3-x+3x^2}$.
4. $\dfrac{1}{1+x+x^3}$.
5. $\dfrac{1}{1-x^3}$.
6. $\dfrac{x^2+3x}{x+2}$.
7. $\dfrac{x^4-3x^2+1}{1+x-x^2}$.
8. $\dfrac{1-x}{5x^2+2x^3}$.
9. $\dfrac{1}{2x^2-6x^3+x^4}$.

Surds.

3. Not every surd, even with only integral powers of x under the radical sign, can be expanded in a series of integral powers of x. If the terms under the sign be arranged in order of ascending powers of x, the first step in the process of extracting the corresponding root will determine when the expansion is possible. That step will also indicate the power of x with which the expansion, when possible, begins.

Thus $\sqrt{(x-2x^2)}$ cannot be expanded in *integral* powers of x, since the first term in any possible expansion must be $x^{\frac{1}{2}}$.

Again, for a similar reason, $\sqrt[3]{(x^2-2x^3+x^4)}$ cannot be expanded into a series of integral powers of x.

Ex. 1. Expand $\sqrt{(1-x^2+2x^3)}$,

in a series, to ascending powers of x.

The first term in the expansion is evidently ± 1. We therefore assume

$$\sqrt{(1-x^2+2x^3)}=1+Bx+Cx^2+Dx^3+Ex^4+\cdots$$

Squaring both sides of the equation, we have

$$1-x^2+2x^3=1+2B\left|\begin{array}{c}x+2C\\+B^2\\ \\ \end{array}\right|\begin{array}{c}x^2+2D\\+2BC\\ \\ \end{array}\left|\begin{array}{c}x^3+2E\\+2BD\\+C^2\end{array}\right|\begin{array}{c}x^4+\cdots.\\ \cdots.\\ \cdots.\end{array}$$

Hence

$$\begin{array}{rl}1=1. & \\ 2B=0, & \text{whence } B=0; \\ 2C+B^2=-1, & \text{whence } C=-\tfrac{1}{2}; \\ 2D+2BC=2, & \text{whence } D=+1; \\ 2E+2BD+C^2=0, & \text{whence } E=-\tfrac{1}{8}; \\ \text{etc.,} & \text{etc.}\end{array}$$

We consequently have

$$\sqrt{(1-x^2+2x^3)}=\pm 1\mp\tfrac{1}{2}x^2\pm x^3\mp\tfrac{1}{8}x^4\pm\cdots.$$

EXERCISES II.

Expand the following expressions in series, to ascending powers of x, to four terms:

1. $\sqrt{(1+x)}$. **2.** $\sqrt{(1-2x^2)}$. **3.** $\sqrt[3]{(1-x^2)}$.
4. $\sqrt{(4-2x+x^2)}$. **5.** $\sqrt{(5+3x+9x^2)}$. **6.** $\sqrt[3]{(1-x+x^2)}$.

§ 3. REVERSION OF SERIES.

1. If one variable be equal to a series of positive integral ascending powers of a second variable, the second variable can be expressed in a series of positive integral ascending powers of the first. This process is called *reversion of series.*

Ex. 1. Revert the series

$$y = x + 2x^2 + 3x^3 + \cdots.$$

Assume
$$x = Ay + By^2 + Cy^3 + \cdots, \tag{1}$$

and substitute in the second member of the last equation the value of y given by the first. Then

$$\begin{aligned} x &= A(x+2x^2+3x^3+\cdots) + B(x+2x^2+3x^3+\cdots)^2 \\ &\quad + C(x+2x^2+3x^3+\cdots)^3 + \cdots \\ &= Ax + \begin{array}{r|}2A\\ +\ B\end{array}\, x^2 + \begin{array}{r|}3A\\ +4B\\ +\ C\end{array}\, x^3 + \cdots. \end{aligned}$$

Hence
$$A = 1.$$

$$2A + B = 0, \quad \text{whence } B = -2;$$

$$3A + 4B + C = 0, \quad \text{whence } C = \ \ 5;$$

$$\text{etc.,} \qquad\qquad \text{etc.}$$

Substituting these values of A, B, C, $\cdots$, in (1), we have

$$x = y - 2y^2 + 5y^3 \cdots.$$

If the series for y in terms of x contain a term free from x, we must find a value of x in a series of powers of y minus that term.

Ex. 2. Revert the series

$$y = 1 + x + x^2 + x^3 + \cdots,$$

or

$$y - 1 = x + x^2 + x^3 + \cdots. \qquad (2)$$

We assume

$$x = A(y-1) + B(y-1)^2 + C(y-1)^3 + \cdots \qquad (3)$$

Substituting in (3) the value of $(y-1)$, given in (2), we obtain

$$x = A(x + x^2 + x^3 + \cdots) + B(x + x^2 + x^3 + \cdots)^2$$
$$+ C(x + x^2 + x^3 + \cdots)^3 + \cdots.$$

Collecting terms containing like powers of x, we have

$$x = Ax + \begin{array}{r|r|l} A & A & \\ +B & +2B & \\ & +C & \end{array} \!\! \begin{array}{l} \end{array}$$

$$x = Ax + \left.\begin{array}{r} A \\ +B \end{array}\right| x^2 + \left.\begin{array}{r} A \\ +2B \\ +C \end{array}\right| x^3 + \cdots.$$

Equating coefficients of like powers of x, we obtain

$$\begin{array}{rll} A = 1, & \text{whence} & A = 1; \\ B + A = 0, & \text{whence} & B = -1; \\ A + 2B + C = 0, & \text{whence} & C = 1. \\ \text{etc.,} & & \text{etc.} \end{array}$$

Substituting these values of the coefficients in the assumed series, we have

$$x = (y-1) - (y-1)^2 + (y-1)^3 - \cdots.$$

EXERCISES III.

Revert each of the following series to four terms:

1. $y = x + x^2 + x^3 + \cdots.$
2. $y = x + 3x^2 + 5x^3 + \cdots.$
3. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots.$
4. $y = 1 - x + 2x^2 - \cdots.$
5. $y = 1 + \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}} + \cdots.$
6. $y = ax + bx^2 + cx^3 + \cdots.$

§ 4. PARTIAL FRACTIONS.

1. It is frequently desirable to separate a rational algebraical fraction into the simpler (*partial*) fractions of which it is the algebraical sum.

E.g.,
$$\frac{2x}{1-x^2}=\frac{1}{1-x}-\frac{1}{1+x}.$$

The process of separating a given fraction into its partial fractions is, therefore, the converse of addition (including subtraction) of fractions; and this fact must guide us in assuming the forms of the partial fractions. We shall separate the given fractions into the simplest partial fractions, that is, fractions which cannot themselves be further decomposed.

We shall also assume that the degree of the numerator is at least one less than that of the denominator. A fraction whose numerator is of a degree equal to or greater than that of its denominator can be first reduced by division to the sum of an integral expression and a fraction satisfying the above condition. The latter fraction will then be decomposed.

The *denominators* of the partial fractions can be definitely assumed. For they are evidently those factors whose lowest common multiple is the denominator of the given fraction. But there is one case of doubt, namely, when a prime factor is repeated in the denominator of the given fraction.

E.g.,
$$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{3}{1-x}+\frac{2}{(1-x)^2}+\frac{1}{1+x};$$
$$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

We could not have decided, in advance, whether either of the two given fractions is the sum of two or of three partial fractions. There must necessarily be a partial fraction having $(1-x)^2$ as a denominator, since, otherwise, the L. C. M. of the denominators would not contain the prime factor $1-x$ to the second power. But it cannot be determined, in advance, whether there is a partial fraction having $1-x$ as a denominator.

In such cases, therefore, it is advisable to make provision for all possible partial fractions by assuming as denominators all repeated factors to the first power, second power, etc.

The numerators of partial fractions thereby assumed, which should not have been included, will acquire the value zero from the subsequent work, so that those fractions drop out of the result.

The *numerators* of the partial fractions must be assumed with undetermined coefficients. Since the numerator of the given fraction is, by the hypothesis, of degree at least one less than the denominator, the same must be true of each partial fraction. We therefore assume, for each numerator, a *complete* rational integral expression with undetermined coefficients of degree one lower than the corresponding denominator.

If any term in the assumed form of the numerator should not have been included, its coefficient will prove to be zero.

An exception to this principle occurs when the denominator of the partial fraction is the second or higher power of a prime factor, as, $(1-x)^2$. In that case the numerator is assumed as it would be according to the above principle if the prime factor occurred to the first power only.

We may briefly restate the above principles:

Separate the denominator of the given fraction into its prime factors. Assume as the denominator of a partial fraction each prime factor; in particular, when a prime factor enters to the nth power, assume that factor to the first power, second power, and so on, to the nth power, as a denominator.

Assume for each numerator a rational integral expression, with undetermined coefficients, of degree one lower than the prime factor in the corresponding denominator.

Let us first decompose the two fractions which we have used to illustrate the theory.

Ex. 1. $$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.$$

Since the prime factor in the denominator of each partial fraction is of the first degree, each numerator is assumed to be of the zeroth degree.

Clearing the equation of fractions, we have

$$\begin{aligned}6-2x^2 &= A(1-x)(1+x)+B(1+x)+C(1-x)^2\\ &= (-A+C)x^2+(B-2C)x+A+B+C.\end{aligned}$$

Since this equation must be true for all values of x, we have

$$\left.\begin{aligned}-A+C&=-2,\\ B-2C&=0,\\ A+B+C&=6.\end{aligned}\right\}\text{ Whence } A=3,\ B=2,\ C=1.$$

Consequently

$$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{3}{1-x}+\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

Ex. **2.** $$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.$$

The forms of the partial fractions are assumed the same as in Ex. 1. We have

$$3+x^2=(-A+C)x^2+(B-2C)x+A+B+C,$$

and then $$\left.\begin{aligned}-A+C&=1,\\ B-2C&=0,\\ A+B+C&=3.\end{aligned}\right\}\text{ Whence } A=0,\ B=2,\ C=1.$$

Therefore $$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

When the factors of the denominator of the given fraction are of the first degree, as in Exx. 1 and 2, the work may be shortened.

Begin with the equation

$$6-2x^2=A(1-x)(1+x)+B(1+x)+C(1-x)^2,$$

of Ex. 1. Since this equation is true for all values of x, we may substitute in it for x any value we please. Let us take such a value as will make one of the prime factors zero.

Substituting 1 for x, we obtain

$$4=2B,\text{ whence } B=2.$$

Next, letting $x = -1$, we have

$$4 = 4C, \text{ whence } C = 1.$$

There is no other value of x which will make a prime factor zero, but any other value, the smaller the better, will give an equation in which we may substitute the values of B and C already obtained.

Letting $x = 0$, we obtain

$$6 = A + B + C, \text{ whence } A = 3.$$

The same method can be applied to Ex. 2.

Ex. 3. $\dfrac{x^2 - x + 3}{x^3 - 1} = \dfrac{x^2 - x + 3}{(x-1)(x^2 + x + 1)} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 + x + 1}.$

In this example the one prime factor being of the second degree we assume the corresponding numerator to be a complete linear expression.

Clearing of fractions, we have

$$x^2 - x + 3 = A(x^2 + x + 1) + (Bx + C)(x - 1) =$$
$$(A + B)x^2 + (A - B + C)x + A - C.$$

Equating coefficients of like powers of x, we obtain

$$A + B = 1, \quad A - B + C = -1, \quad A - C = 3;$$

whence, $\qquad A = 1, \; B = 0, \; C = -2.$

Or, we might have used the second method, beginning with

$$x^2 - x + 3 = A(x^2 + x + 1) + (Bx + C)(x - 1).$$

Letting $x = 1$, we obtain

$$3 = 3A, \text{ whence } A = 1.$$

Since no other value of x will make a factor vanish, we take any simple values. When $x = 0$, we have

$$3 = A - C, \text{ whence } C = -2.$$

Finally, letting $x = -1$, we have

$$5 = A + 2B - 2C, \text{ whence } B = 0.$$

Ex. 4.

$$\frac{2 - 2x + 4x^2}{(1 + x^2)^2(1 - x)} = \frac{Ax + B}{1 + x^2} + \frac{Cx + D}{(1 + x^2)^2} + \frac{E}{1 - x}.$$

The prime factors in the denominators of the first two partial fractions being of the second degree, expressions of the first degree are assumed as numerators.

Clearing of fractions, we have

$$\begin{aligned}2-2x+4x^2 &= (Ax+B)(1+x^2)(1-x)+(Cx+D)(1-x)+E(1+x^2)^2\\ &= (-A+E)x^4+(A-B)x^3+(-A+B-C+2E)x^2\\ &\qquad +(A-B+C-D)x+(B+D+E).\end{aligned}$$

Equating coefficients of like powers of x, we obtain

$$-A+E=0,\ A-B=0,\ -A+B-C+2E=4,$$
$$A-B+C-D=-2,\ B+D+E=2;$$

whence $\quad A=1,\ B=1,\ C=-2,\ D=0,\ E=1.$

EXERCISES IV.

Separate the following fractions into partial fractions:

1. $\frac{1}{7x-x^2-12}$.

2. $\frac{6x}{x^2-4}$.

3. $\frac{x^2+2x-1}{x^2-1}$.

4. $\frac{x^2}{x^2-4}$.

5. $\frac{5}{1-x^2}$.

6. $\frac{7x^2+19x}{x^2-9}$.

7. $\frac{2x^2+3x-1}{x^3-x}$.

8. $\frac{1+x}{9-x^2}$.

9. $\frac{3x^2+1}{(x+1)(x-1)^2}$.

10. $\frac{4x}{x^2-1}$.

11. $\frac{x^2+5x+10}{(x+1)(x+2)(x+3)}$.

12. $\frac{5x(x+3)}{(2x+1)(2x-1)(x+1)}$.

13. $\frac{3-x}{(2x+1)(2x+3)(x-1)}$.

14. $\frac{x^2+90x-9}{6(x^2-9)(x-3)}$.

15. $\frac{3x+2}{(x^2-1)(x-2)}$.

16. $\frac{x}{(x-1)^3}$.

17. $\frac{x+1}{(x-1)^4}$.

18. $\frac{1}{x^3-1}$.

19. $\frac{1}{x^3+1}$.

20. $\frac{x+1}{x^3-1}$.

21. $\frac{x-1}{x^3+1}$.

22. $\frac{1}{x^4-1}$.

23. $\frac{1}{x^2(x^2+1)}$.

24. $\frac{x^2-15x-18}{(x^2-9)(x-1)}$.

25. $\frac{44-9x}{x^3+64}$.

26. $\frac{1}{x^4+x}$.

27. $\frac{x}{x^3-1}$.

CHAPTER XXXIV.

CONTINUED FRACTIONS.

1. If the numerator and denominator of $\frac{30}{43}$ be divided by the numerator, we have

$$\frac{30}{43} = \frac{30 \div 30}{43 \div 30} = \frac{1}{1 + \frac{13}{30}}.$$

Reducing $\frac{13}{30}$, and subsequent fractions, in a similar way, we obtain

$$\frac{30}{43} = \frac{1}{1 + \dfrac{13 \div 13}{30 \div 13}} = \frac{1}{1 + \dfrac{1}{2 + \dfrac{4}{13}}} = \frac{1}{1 + \dfrac{1}{2 + \dfrac{1}{3 + \dfrac{1}{4}}}}.$$

The complex fraction thus obtained is usually written more compactly thus:

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}.$$

Observe that in the last form the signs + are written on a line with the denominators to distinguish the complex fraction from the sum of common fractions. It is important to keep in mind that in both forms the numerator at any stage is the numerator of a fraction whose denominator is the entire complex fraction which is written below and to the right of that particular numerator.

2. A Continued Fraction is a fraction whose numerator is an integer, and whose denominator is an integer plus a fraction whose numerator is an integer, and whose denominator is an integer plus a fraction, etc.

A continued fraction frequently occurs in connection with an integral term.

E.g.,
$$5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{+3\frac{1}{4}}}}=5+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}.$$

In such cases it is customary to call the entire mixed number the continued fraction.

The general form of a continued fraction, therefore, is:

$$n+\cfrac{n_1}{d_1+\cfrac{n_2}{d_2+\cfrac{n_3}{d_3+\cdots}}}=n+\frac{n_1}{d_1+}\,\frac{n_2}{d_2+}\,\frac{n_3}{d_3+}\cdots.$$

3. We shall confine ourselves in this chapter to continued fractions in which the numerators are all 1, and the denominators all positive integers; of the general form, therefore,

$$n+\frac{1}{d_1+}\,\frac{1}{d_2+}\,\frac{1}{d_3+}\cdots,$$

in which the d's are all positive integers, and n is a positive integer or 0.

The n and the d's are called **Partial Quotients.**

4. A **Terminating,** or **Finite Continued Fraction,** is one in which the number of partial quotients is limited, as in the example given above.

A **Non-terminating,** or **Infinite Continued Fraction,** is one in which the number of partial quotients is unlimited or infinite.

To Convert a Common Fraction into a Terminating Continued Fraction.

5. Compare the work in Art. 1 of reducing $\frac{30}{43}$ to a continued fraction, with the work of finding the G. C. M. of 30 and 43:

```
30)43(1
   30
   13)30(2
      26
       4)13(3
         12
          1)4(4
            4
```

Observe that the successive *quotients* in the latter process are the partial quotients of the continued fraction. This is as it should be, since a comparison of the two processes shows that the successive steps of division in getting the partial quotients are identical with those in finding the G. C. M.

The method is evidently perfectly general and may be applied to any common fraction. If the fraction be improper, the first quotient will be the integral part of the continued fraction, and the remaining quotients the successive partial quotients of the continued fraction proper.

Ex. Reduce $\frac{151}{45}$ to a continued fraction.

By the method of G. C. M., we have

```
45)151(3
   135
    16)45(2
       32
       13)16(1
          13
           3)13(4
             12
              1)3(3
                3
```

Therefore, $$\frac{151}{45} = 3 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3}.$$

To Reduce a Terminating Continued Fraction to a Common Fraction.

6. We have only to retrace the steps taken in the preceding article in forming a continued fraction from a common fraction. Thus,

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{4}{13}=\frac{1}{1+}\,\frac{13}{30}=\frac{30}{43}.$$

Evidently this method is also perfectly general.

We therefore conclude that any common fraction can be converted into a terminating continued fraction, and, conversely, that any terminating continued fraction can be reduced to a common fraction.

The latter reduction becomes laborious in the case of a continued fraction with many partial quotients, and a simpler method will now be given.

7. A Convergent of a continued fraction is that part of it obtained by stopping with a definite partial quotient.

Thus, in $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4},$$

the first convergent is $\frac{1}{1}$; the second is $\frac{1}{1+}\,\frac{1}{2}, =\frac{2}{3}$;

the third is $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}, =\frac{7}{10};$$

the fourth is $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}, =\frac{30}{43}.$$

For convenience, we will call the integral term, when there is one, the *zeroth* convergent, so that the nth convergent will always end with the nth partial quotient. We will denote the successive convergents by $\frac{N_0}{D_0}, \frac{N_1}{D_1}, \frac{N_2}{D_2}$, etc.

The above convergents may then be written thus:

$$\frac{N_1}{D_1}=\frac{1}{1},\ \frac{N_2}{D_2}=\frac{2}{3};$$

$$\frac{N_3}{D_3}=\frac{7}{10}=\frac{3\times 2+1}{3\times 3+1}=\frac{3\times N_2+N_1}{3\times D_2+D_1};$$

$$\frac{N_4}{D_4}=\frac{30}{43}=\frac{4\times\ 7+2}{4\times 10+3}=\frac{4\times N_3+N_2}{4\times D_3+D_2}.$$

That is, to form the numerator of the third convergent, multiply the numerator of the second by the third partial quotient, and to the product add the numerator of the first convergent. To form the numerator of the fourth convergent, multiply the numerator of the third convergent by the fourth partial quotient, and to the product add the numerator of the second convergent. In like manner, form the denominators from the denominators of the two preceding convergents.

In general,

The numerator of any convergent after the second (after the first if there be a zeroth convergent) is formed by multiplying the numerator of the immediately preceding convergent by that partial quotient with which the convergent to be computed ends, and to the product adding the numerator of the second preceding convergent; the denominator of the same convergent is formed in like manner from the denominators of the two convergents immediately preceding.

The principle holds for the second convergent when there is a zeroth convergent; and in all cases for the third convergent.

For
$$\frac{N_0}{D_0}=\frac{n}{1},\ \frac{N_1}{D_1}=n+\frac{1}{d_1}=\frac{nd_1+1}{d_1},$$

$$\frac{N_2}{D_2}=n+\frac{1}{d_1+}\,\frac{1}{d_2}=n+\frac{d_2}{d_1d_2+1}=\frac{nd_1d_2+n+d_2}{d_1d_2+1}$$

$$=\frac{d_2(nd_1+1)+n}{d_1d_2+1}=\frac{d_2N_1+N_0}{d_2D_1+D_0},$$

$$\frac{N_3}{D_3} = n + \frac{1}{d_1+}\,\frac{1}{d_2+}\,\frac{1}{d_3} = n + \frac{1}{d_1+}\,\frac{d_3}{d_2d_3+1} = n + \frac{d_2d_3+1}{d_1d_2d_3+d_1+d_3}$$

$$= \frac{nd_1d_2d_3 + nd_1 + nd_3 + d_2d_3 + 1}{d_1d_2d_3 + d_1 + d_3}$$

$$= \frac{d_3(nd_1d_2 + n + d_2) + (nd_1 + 1)}{d_3(d_1d_2 + 1) + d_1} = \frac{d_3N_2 + N_1}{d_3D_2 + D_1}.$$

If the principle holds up to and including any convergent, it holds for the next convergent.

Suppose it holds up to and including the *k*th convergent. We then have

$$\frac{N_k}{D_k} = d + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cdots \cfrac{1}{d_k}}} = \frac{d_kN_{k-1} + N_{k-2}}{d_kD_{k-1} + D_{k-2}}.$$

Now
$$\frac{N_{k+1}}{D_{k+1}} = d + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cdots \cfrac{1}{d_k + \cfrac{1}{d_{k+1}}}}}$$

differs from the preceding convergent only in having

$$d_k + \frac{1}{d_{k+1}}$$

as a denominator where the preceding has d_k. Therefore, if we substitute

$$d_k + \frac{1}{d_{k+1}} \text{ for } d_k \text{ in } \frac{d_kN_{k-1} + N_{k-2}}{d_kD_{k-1} + D_{k-2}},$$

we obtain an expression for $\dfrac{N_{k+1}}{D_{k+1}}$,

without assuming that the principle holds beyond the *k*th convergent.

Consequently,

$$\frac{N_{k+1}}{D_{k+1}} = \frac{\left(d_k + \dfrac{1}{d_{k+1}}\right)N_{k-1} + N_{k-2}}{\left(d_k + \dfrac{1}{d_{k+1}}\right)D_{k-1} + D_{k-2}} = \frac{d_kd_{k+1}N_{k-1} + N_{k-1} + d_{k+1}N_{k-2}}{d_kd_{k+1}D_{k-1} + D_{k-1} + d_{k+1}D_{k-2}}$$

$$= \frac{d_{k+1}(d_kN_{k-1} + N_{k-2}) + N_{k-1}}{d_{k+1}(d_kD_{k-1} + D_{k-2}) + D_{k-1}} = \frac{d_{k+1}N_k + N_{k-1}}{d_{k+1}D_k + D_{k-1}},$$

a result in accordance with the principle.

Therefore, since the principle holds to and including the third convergent, it holds for the fourth; then, since it holds for the fourth, it holds for the fifth; and so on.

This method of proof is called **Proof by Mathematical Induction.**

Properties of Convergents.

8. (i.) *The successive convergents, beginning with the zeroth, are alternately less and greater than the continued fraction.*

Thus, from
$$\frac{151}{45}=3+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3},$$

we have
$$\frac{N_0}{D_0}=\frac{3}{1},\ \frac{N_1}{D_1}=\frac{7}{2},\ \frac{N_2}{D_2}=\frac{10}{3},\ \frac{N_3}{D_3}=\frac{47}{14},\ \frac{N_4}{D_4}=\frac{151}{45},$$

and
$$\frac{3}{1}<\frac{151}{45},\ \frac{7}{2}>\frac{151}{45},\ \frac{10}{3}<\frac{151}{45},\ \frac{47}{14}>\frac{151}{45}.$$

The symbol $\sim$, read *difference between*, is placed between two numbers to indicate that the less is to be subtracted from the greater. *E.g.*, $3\sim4=4\sim3=4-3=1$.

(ii.) *The difference between any two consecutive convergents is* 1 *divided by the product of their denominators.*

Thus,
$$\frac{3}{1}\sim\frac{7}{2}=\frac{1}{1\times2},\ \frac{7}{2}\sim\frac{10}{3}=\frac{1}{2\times3},\ \frac{10}{3}\sim\frac{47}{14}=\frac{1}{3\times14},\ \text{etc.}$$

(iii.) *Each convergent is nearer in value to the continued fraction than any preceding convergent.*

Thus,
$$\frac{151}{45}\sim\frac{3}{1}=\frac{16}{45};\ \ \frac{151}{45}\sim\frac{7}{2}=\frac{13}{90};\ \ \frac{151}{45}\sim\frac{10}{3}=\frac{3}{135};$$

and
$$\frac{16}{45}>\frac{13}{90}>\frac{3}{135}.$$

(iv.) *The convergents of even order continually increase, but are always less than the continued fraction; while the convergents of odd order continually decrease, but are always greater than the continued fraction.*

E.g.,
$$\frac{3}{1}<\frac{10}{3}<\frac{151}{45};\ \ \frac{7}{2}>\frac{47}{14}.$$

In a terminating continued fraction, the last convergent will, of course, be the continued fraction, and therefore neither greater nor less than itself.

The proofs follow:

Let $$V = n + \frac{1}{d_1+}\,\frac{1}{d_2+}\,\frac{1}{d_3+}\cdots.$$

(i.) The zeroth convergent is too small by

$$\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots.$$

In the first convergent, the partial quotient d_1 is too small by $\frac{1}{d_2+}\cdots$; hence $\frac{1}{d_1}$ is too great, and therefore $n+\frac{1}{d_1}$ is too great.

In the second convergent, the second partial quotient d_2 is too small by $\frac{1}{d_3+}$; hence $\frac{1}{d_2}$ is too great, and therefore $d_1+\frac{1}{d_2}$ is also too great; finally $\frac{1}{d_1+}\,\frac{1}{d_2}$ is too small, and $n+\frac{1}{d_1+}\,\frac{1}{d_2}$ is too small.

And so on.

(ii.) Since $$\frac{N_k}{D_k} \sim \frac{N_{k+1}}{D_{k+1}} = \frac{N_kD_{k+1} \sim D_kN_{k+1}}{D_kD_{k+1}},$$

we have only to prove

$$N_kD_{k+1} \sim D_kN_{k+1} = 1.$$

The law holds for the first two convergents.

For $$\frac{N_0}{D_0} \sim \frac{N_1}{D_1} = \frac{n}{1} \sim \frac{nd_1+1}{d_1} = \frac{nd_1 \sim (nd_1+1)}{1 \cdot d_1} = \frac{1}{d_1}.$$

If it holds for any two consecutive convergents, it holds for the second of these two and the next convergent.

We have

$$\begin{aligned} N_kD_{k+1} \sim D_kN_{k+1} &= N_k(d_{k+1}D_k + D_{k-1}) \sim D_k(N_kd_{k+1} + N_{k-1}) \\ &= N_kD_{k-1} \sim D_kN_{k-1}. \end{aligned}$$

Therefore, if the principle holds for

$$\frac{N_{k-1}}{D_{k-1}} \sim \frac{N_k}{D_k},$$

it holds for

$$\frac{N_k}{D_k} \sim \frac{N_{k+1}}{D_{k+1}}.$$

(iii.) $\frac{N_{k+1}}{D_{k+1}}$ differs from V only in having d_{k+1} where V has

$$d_{k+1}+\frac{1}{d_{k+2}}+\cdots, \; = K, \text{ say.}$$

Then, $$\frac{N_k}{D_k}=\frac{d_kN_{k-1}+N_{k-2}}{d_kD_{k-1}+D_{k-2}}, \quad V=\frac{KN_k+N_{k-1}}{KD_k+D_{k-1}}.$$

But $$\frac{N_k}{D_k}\sim V=\frac{N_k}{D_k}\sim\frac{KN_k+N_{k-1}}{KD_k+D_{k-1}}=\frac{N_kD_{k-1}\sim N_{k-1}D_k}{D_k(KD_k+D_{k-1})}$$

$$=\frac{1}{D_k(KD_k+D_{k-1})},$$

and $$\frac{N_{k-1}}{D_{k-1}}\sim V=\frac{N_{k-1}}{D_{k-1}}\sim\frac{KN_k+N_{k-1}}{KD_k+D_{k-1}}=\frac{K(N_{k-1}D_k\sim N_kD_{k-1})}{D_{k-1}(KD_k+D_{k-1})}$$

$$=\frac{K}{D_{k-1}(KD_k+D_{k-1})}.$$

But $K>1$ and $D_{k-1}<D_k$. Therefore,

$$\frac{N_k}{D_k}\sim V<\frac{N_{k-1}}{D_{k-1}}\sim V.$$

Hence, any convergent is nearer in value to the continued fraction than the immediately preceding convergent, and consequently than *any* preceding convergent.

(iv.) The proof follows at once from (iii.) and (i.).

Limit to Error of Any Convergent.

9. Since, by Art. 8 (i.), the value of a continued fraction is between the values of any two consecutive convergents, it must differ from either of them by less than they differ from each other.

Therefore, an error of taking $\frac{N_k}{D_k}$ for the continued fraction is, by Art. 8 (ii.), less than $\frac{1}{D_kD_{k+1}}$.

But $$D_{k+1}=d_{k+1}D_k+D_{k-1}>d_{k+1}D_k.$$

Therefore, the error of $\frac{N_k}{D_k}$ is less than $\frac{1}{d_{k+1}D_k^2}$.

Hence, to find a convergent which differs from the continued fraction by less than $\frac{1}{m}$, we have only to compute successive convergents up to $\frac{N_k}{D_k}$, wherein $d_{k+1}D_k^2 < m$.

Ex. Find an approximation to 3.14159, correct to five decimal places.

We have

$$3.14159 = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4}.$$

The successive convergents are $\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \cdots$.

The error of $\frac{355}{113}$ is less than $\frac{1}{25(113)^2}$ and with greater reason less than $\frac{1}{25(100)^2}$, $= .000004$.

Therefore $\frac{355}{113}$ is the required approximation.

To Reduce a Quadratic Surd to a Continued Fraction.

10. The general method may be illustrated by particular examples.

Ex. Reduce $\sqrt{14}$ to a continued fraction.

Since the greatest integer contained in $\sqrt{14}$ is 3, we assume

$$\sqrt{14} = 3 + \frac{1}{d_1}.$$

Then $d_1 = \frac{1}{\sqrt{14}-3} = \frac{\sqrt{14}+3}{14-9} = \frac{\sqrt{14}+3}{5}$.

Since the greatest integer in this value of d_1 is 1, we assume

$$d_1 = \frac{\sqrt{14}+3}{5} = 1 + \frac{1}{d_2}.$$

Then $d_2 = \frac{5}{\sqrt{14}-2} = \frac{5(\sqrt{14}+2)}{10} = \frac{\sqrt{14}+2}{2}$.

In like manner, we assume

$$d_2 = \frac{\sqrt{14}+2}{2} = 2 + \frac{1}{d_3}.$$

Then $d_3 = \dfrac{2}{\sqrt{14}-2} = \dfrac{2(\sqrt{14}+2)}{10} = \dfrac{\sqrt{14}+2}{5} = 1 + \dfrac{1}{d_4}.$

Similarly,

$$d_4 = \frac{5}{\sqrt{14}-3} = \frac{5(\sqrt{14}+3)}{5} = \sqrt{14}+3 = 6 + \frac{1}{d_5};$$

$$d_5 = \frac{1}{\sqrt{14}-3}, \text{ etc.}$$

Since this process may be continued indefinitely, we obtain an infinite continued fraction by substituting, in succession, the values obtained for d_1, d_2, d_3, $\cdots$.

We then have

$$\sqrt{14} = 3 + \frac{1}{d_1} = 3 + \frac{1}{1+}\frac{1}{d_2} = 3 + \frac{1}{1+}\frac{1}{2+}\frac{1}{d_3}$$

$$= 3 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\cdots.$$

Observe that the value obtained for

$$d_5 = \frac{1}{\sqrt{14}-3}$$

is the same as that for d_1, so that

$$d_6 = d_2,\ d_7 = d_3,\ d_8 = d_4,\ d_9 = d_5 = d_1, \text{ etc.}$$

Therefore the partial quotients 1, 2, 1, 6, are repeated indefinitely.

11. **A Periodic Continued Fraction** is an infinite continued fraction in which the partial quotients are repeated in sets of one or more.

Ex. Reduce $\dfrac{5-\sqrt{3}}{6}$ to a periodic continued fraction.

Since $\dfrac{5-\sqrt{3}}{6} < 1$, we assume $\dfrac{5-\sqrt{3}}{6} = \dfrac{1}{d_1}.$

Then $d_1 = \dfrac{6}{5-\sqrt{3}} = \dfrac{3(5+\sqrt{3})}{11} = 1 + \dfrac{1}{d_2}.$

And $$d_2 = \frac{11}{4+3\sqrt{3}} = \frac{11(4-3\sqrt{3})}{-11} = \frac{3\sqrt{3}-4}{1} = 1 + \frac{1}{d_3}.$$

Similarly, $$d_3 = \frac{1}{3\sqrt{3}-5} = \frac{3\sqrt{3}+5}{2} = 5 + \frac{1}{d_4}.$$

Likewise, $$d_4 = \frac{2}{3\sqrt{3}-5} = 3\sqrt{3}+5 = 10 + \frac{1}{d_5}.$$

Finally, $$d_5 = \frac{1}{3\sqrt{3}-5} = d_3.$$

Therefore, only the third and fourth partial quotients are repeated, and the required fraction is

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{10+}\,\frac{1}{5+}\,\frac{1}{10+}\cdots.$$

Application of Convergents.

12. It is often convenient to substitute for a fraction with large terms, or for a quadratic surd, a convergent with comparatively small terms, provided that convergent approximates closely enough to the true value.

Ex. 1. $$\tfrac{351}{1003} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{50}}}}$$

By Art. 9, we should expect the third convergent to be a close approximation, since the following partial quotient, 50, is large.

We have $$\frac{N_1}{D_1} = \frac{1}{2},\ \frac{N_2}{D_2} = \frac{1}{3},\ \frac{N_3}{D_3} = \frac{7}{20}.$$

Therefore, by Art. 9, the error of the third convergent is less than

$$\frac{1}{50 \times 20^2},\ = .00005.$$

Consequently, $\tfrac{7}{20}$ represents the true value of $\tfrac{351}{1003}$ correctly to four decimal places.

Ex. 2. Given $\sqrt{14} = 3 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$, find the error of the seventh convergent.

The student may satisfy himself that $\frac{N_7}{D_7} = \frac{449}{120}$.

The error of $\frac{N_7}{D_7} < \frac{1}{6(120)^2} < \frac{1}{86400} < .000011\cdots$.

Therefore $\frac{449}{120}$ is correct to four decimal places.

To Reduce a Periodic Continued Fraction to an Irrational Number.

13. We will take as an example the result of Ex. Art. 10.

Assume $$x = 3 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\cdots,$$

then $$x - 3 = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6\boldsymbol{+}}\cdots.$$

Since the partial quotients 1, 2, 1, 6 are repeated indefinitely in that order, the continued fraction whose first partial quotient is the first periodic number (*i.e.*, 1) at any stage, and which is continued indefinitely, differs in no respect from the given periodic continued fraction. For example, the periodic continued fraction which follows the heavy plus sign $(\boldsymbol{+})$, in the value of $x - 3$ above, is the same as the entire continued fraction, which is the value of $x - 3$.

We may therefore substitute $x - 3$ for the part of the continued fraction which follows that particular plus sign. We thus have

$$x - 3 = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+x-3} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+x} = \frac{11+3x}{15+4x}.$$

From this equation we obtain

$$4x^2 = 56, \text{ or } x = \sqrt{14}.$$

14. If the continued fraction be not periodic from the beginning, we first reduce the periodic part by itself as above, and substitute its value in the given continued fraction. The latter is then a terminating continued fraction and can be reduced to a simple fraction, whose numerator and denominator will not, however, be rational.

Ex. $$x = \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{3+}\,\frac{1}{5+}\cdots,$$

the periodic part commencing with the third partial quotient.

Assume $$y = \frac{1}{3+}\,\frac{1}{5+}\cdots = \frac{1}{3+}\,\frac{1}{5+y} = \frac{5+y}{16+3y};$$

hence $$3y^2 + 16y = 5 + y,$$

and $$y = \frac{-15+\sqrt{285}}{6}.\quad \text{But } x = \frac{1}{2+}\,\frac{1}{1+y} = \frac{77-\sqrt{285}}{166}.$$

EXERCISES.

Compute the successive convergents to

1. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3}.$ **2.** $2 + \frac{1}{5+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4}.$

Reduce each of the following fractions to a continued fraction, find its convergents, and determine a limit to the error of the third convergent.

3. $\frac{25}{39}$. **4.** $\frac{14}{105}$. **5.** $\frac{85}{16}$. **6.** $\frac{253}{91}$. **7.** $\frac{819}{37}$.
8. $\frac{532}{1193}$. **9.** $\frac{2771}{578}$. **10.** $27\frac{43}{51}$. **11.** .4751. **12.** 5.0372.

Reduce each of the following surds to continued fractions, find the first five convergents, and determine a limit to the error of the fourth convergent.

13. $\sqrt{7}$. **14.** $\sqrt{23}$. **15.** $\sqrt{2.5}$. **16.** $\sqrt{29}$. **17.** $2\sqrt{45}$.
18. $\frac{1+\sqrt{2}}{5}$. **19.** $\frac{22-\sqrt{7}}{4}$. **20.** $\frac{2+\sqrt{3}}{2-\sqrt{3}}$. **21.** $\frac{11+\sqrt{7}}{5}$.

Reduce each of the following periodic continued fractions to a surd:

22. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$

23. $3 + \frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\cdots.$

24. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{7+}\,\frac{1}{3+}\,\frac{1}{7+}\,\frac{1}{3+}\cdots.$

25. Express the decimal .43429, which will occur in the next chapter, as a continued fraction, find its fifth convergent, and determine the limit to the error of this convergent.

26. Express the decimal 2.71828, which will occur in the next chapter, as a continued fraction, find its seventh convergent, and determine a limit to the error of this convergent.

27. The true length of the equinoctial year is $365^d\,5^h\,48^m\,46^s$. Reduce the ratio $5^h\,48^m\,46^s : 24^h$, to a continued fraction, and hence show how often leap year should come.

CHAPTER XXXV.

LOGARITHMS.

1. An equation of the form $b^n = a$, in which it is required to find one of the three numbers, a, b, n, in terms of the two other numbers, supposed to be known, leads to three different operations.

(i.) Given b and n, to find a. We then have $a = b^n$. Expressions of the form b^n have been considered in the preceding chapters.

(ii.) Given a and n, to find b. We then have $b = \sqrt[n]{a}$. Expressions of the form $\sqrt[n]{a}$ have already been considered.

(iii.) Given a and b, to find n. Designating the unknown number by x, we have

$$b^x = a.$$

This relation, wherein a and b are *real* and *positive*, forms the subject-matter of this chapter. It will be proved later that a value of x can always be found to satisfy the given equation.

This value of x is frequently an irrational number. We must therefore prove that the principles established in Ch. XXVI. for rational powers hold also for irrational powers.

Irrational Powers.

2. An **Irrational Power** is a power whose exponent is an irrational number; as $x^{\sqrt{2}}$.

3. Let b be any real positive number greater than 1, and I be a positive irrational number defined by the relation (Ch. XVIII., Art. 8):

$$\frac{m}{n} < I < \frac{m+1}{n}.$$

Then the two *rational* powers $b^{\frac{m}{n}}$ and $b^{\frac{m+1}{n}}$ have the properties (i.) and (ii.), Art. 6, Ch. XVIII.

Since $\frac{m}{n}$ increases and $\frac{m+1}{n}$ decreases as n increases, it follows from Ch. XVII., Art. 8 (i.) and (ii.) and Art. 7 (i.), that $b^{\frac{m}{n}}$ increases and $b^{\frac{m+1}{n}}$ decreases as n increases, and that $b^{\frac{m}{n}} < b^{\frac{m+1}{n}}$.

The difference $b^{\frac{m+1}{n}} - b^{\frac{m}{n}} = b^{\frac{m}{n}}(b^{\frac{1}{n}} - 1)$ is positive, and can be made less than any assigned number, however small.

For, let
$$b^{\frac{1}{n}} - 1 = d, \qquad (1)$$
wherein d is positive, since $b^{\frac{1}{n}} > 1$, by Ch. XVII., Art. 8 (ii.).

We are then to prove that d can be made less than any assigned number, however small, by increasing n indefinitely.

From (1), we have $b^{\frac{1}{n}} = 1 + d$, or $b = (1 + d)^n$.

By Ch. XVII., Art. 15, $1 + nd < (1 + d)^n$.

Therefore
$$1 + nd < b, \text{ or } d < \frac{b-1}{n}.$$

Consequently, as n increases indefinitely, $\frac{b-1}{n}$, and hence also d, decreases indefinitely, and can be made less than any assigned number.

But $b^{\frac{m}{n}}, < b^{\frac{m+1}{n}}$, is less than some definite finite number R. Therefore, $b^{\frac{m+1}{n}} - b^{\frac{m}{n}} < \frac{R(b-1)}{n}$.

By increasing n beyond any assigned number, however great, $\frac{R(b-1)}{n}$ can be made less than any assigned number, however small.

Therefore the two series of powers $b^{\frac{m}{n}}$ and $b^{\frac{m+1}{n}}$ determine a positive number which lies between them. This number is defined as a^I. That is,
$$b^{\frac{m}{n}} < b^I < b^{\frac{m+1}{n}}.$$

4. In like manner it can be shown that if $-I$ be an irrational number, defined by the relation
$$-\frac{m+1}{n} < -I < -\frac{m}{n},$$
then the two series of powers, $b^{-\frac{m+1}{n}}$ and $b^{-\frac{m}{n}}$, determine a positive number which lies between them. This number is defined as a^{-I}. That is,
$$b^{-\frac{m+1}{n}} < b^{-I} < b^{-\frac{m}{n}}.$$

5. It follows directly from the definition of b^{-I}, that $b^{-I} = \frac{1}{a^I}$.

6. It can now be proved that the principles of rational powers hold also for irrational powers.

Let b^{I_1} and b^{I_2} be two irrational powers defined by the relations

$$b^{\frac{m_1}{n_1}} < b^{I_1} < b^{\frac{m_1+1}{n_1}}, \quad b^{\frac{m_2}{n_2}} < b^{I_2} < b^{\frac{m_2+1}{n_2}}.$$

(i.) If the corresponding rational powers of the series which define b^{I_1} and b^{I_2} be multiplied, we obtain the two series of numbers

$$b^{\frac{m_1}{n_1}} b^{\frac{m_2}{n_2}} \text{ and } b^{\frac{m_1+1}{n_1}} b^{\frac{m_2+1}{n_2}}.$$

The numbers of these series have the properties (i.) and (ii.), Art. 6, Ch. XVIII. The proof is similar to that given in Ch. XVIII., Art. 15.

Therefore the two series

$$b^{\frac{m_1}{n_1}} b^{\frac{m_2}{n_2}} \text{ and } b^{\frac{m_1+1}{n_1}} b^{\frac{m_2+1}{n_2}}$$

determine a positive number which lies between them. This number is defined as the product $b^{I_1}b^{I_2}$.

That is, $$b^{\frac{m_1}{n_1}} b^{\frac{m_2}{n_2}} < b^{I_1}b^{I_2} < b^{\frac{m_1+1}{n_1}} b^{\frac{m_2+1}{n_2}}.$$

In like manner it can be shown that the two series

$$b^{\frac{m_1}{n_1}+\frac{m_2}{n_2}} \text{ and } b^{\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}}$$

determine a positive number which lies between them. This number is defined as $b^{I_1+I_2}$.

That is, $$b^{\frac{m_1}{n_1}+\frac{m_2}{n_2}} < b^{I_1+I_2} < b^{\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}}.$$

But since $$b^{\frac{m_1}{n_1}} b^{\frac{m_2}{n_2}} = b^{\frac{m_1}{n_1}+\frac{m_2}{n_2}} \text{ and } b^{\frac{m_1+1}{n_1}} b^{\frac{m_2+1}{n_2}} = b^{\frac{m_1+1}{n_1}+\frac{m_2+1}{n_2}},$$

the two numbers $b^{I_1+I_2}$ and $b^{I_1}b^{I_2}$ are determined by the same relation, and are therefore equal.

That is, $$b^{I_1}b^{I_2} = b^{I_1+I_2}.$$

In a similar manner the principle can be proved when the exponents, either or both, are negative irrational numbers.

(ii.) We have $$\frac{b^{I_1}}{b^{I_2}} = b^{I_1}b^{-I_2} = b^{I_1+(-I_2)} = b^{I_1-I_2},$$

wherein I_1 and I_2, either or both, are negative.

Principles (III.)-(V.), Ch. XXVI., can be proved in a similar manner.

7. In the proofs of the preceding principles, the base was assumed to be greater than 1. Similar reasoning will, however, apply when the base is less than 1 and positive.

For, if $b < 1$ and positive, it can be shown that the irrational power is defined by the relation

$$b^{\frac{m+1}{n}} < b^I < b^{\frac{m}{n}}.$$

Solution of the Equation $b^x = a$.

8. (i.) First, let $b > 1$. The powers

$$\cdots,\ b^{-2},\ b^{-1},\ b^0,\ b^1,\ b^2,\ \cdots,$$

increase toward the right beyond any positive number, however great, as the exponents increase without limit. For, by Ch. XVII., Art. 15,

$$(1 + d)^n > 1 + nd,$$

wherein n and d are positive. We can make $1 + nd$ greater than any assigned number, however great, by increasing n without limit. But $1 + d$ represents *any number greater than* 1.

The same series decreases toward the left below any assigned positive number, however small, as the absolute values of the exponents increase without limit, by Ch. XXVII., § 3, Art. 8. For $b^{-n} = \left(\frac{1}{b}\right)^n$, and $\frac{1}{b} < 1$.

Then a will either be equal to one of these powers or lie between two consecutive powers. In the former case, x is equal to a positive or a negative integer. In the latter case, x lies between two consecutive numbers of the series

$$\cdots -2,\ -1,\ 0,\ +1,\ +2,\ \cdots.$$

Let b^k and b^{k+1} be the two powers between which a is found to lie; *i.e.*, $b^k < a < b^{k+1}$, wherein k is 0, or any positive or negative integer.

Then x lies between k and $k + 1$; or, $k < x < k + 1$.

The interval between $k + 1$ and k, $= 1$, we now divide into ten equal parts, and form the series of powers,

$$b^k,\ b^{k+\frac{1}{10}},\ b^{k+\frac{2}{10}},\ \cdots,\ b^{k+\frac{9}{10}},\ b^{k+1}.$$

Then a, which lies between b^k and b^{k+1}, must either be equal to one of these powers, or lie between two consecutive powers.

In the former case, x is equal to a fraction $k + \frac{k_1}{10}$, wherein k_1 is one of the numbers $1, \cdots, 9$.

In the latter case, let $b^{k+\frac{k_1}{10}}$ and $b^{k+\frac{k_1+1}{10}}$ be the two powers between which a is found to lie; that is,

$$b^{k+\frac{k_1}{10}} < a < b^{k+\frac{k_1+1}{10}}.$$

Wherein k is one of the numbers 0, 1, ⋯ 9. Then x lies between

$$k+\frac{k_1}{10} \text{ and } k+\frac{k_1+1}{10};$$

or,

$$k+\frac{k_1}{10}<x<k+\frac{k_1+1}{10}.$$

By continuing the process, we can prove that a either is equal to a power of the form $b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}}$, or lies between two consecutive powers $b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}}$, and $b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}}$.

In the former case, $a=b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}}$, and therefore

$$x=k+\frac{k_1}{10}+\frac{k_2}{10}+\cdots+\frac{k_p}{10^p},$$

a *rational* number.

In the latter case,

$$b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}}<a<b^{k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}}; \qquad (1)$$

and therefore,

$$k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p}<x<k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p}. \qquad (2)$$

As in Ch. XVIII., Art. 7, we may designate

$$k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p}{10^p} \text{ by } \frac{m}{n},$$

and

$$k+\frac{k_1}{10}+\frac{k_2}{10^2}+\cdots+\frac{k_p+1}{10^p} \text{ by } \frac{m+1}{n}.$$

The relations (1) and (2) then become

$$b^{\frac{m}{n}}<a<b^{\frac{m+1}{n}} \quad (3), \quad \text{and} \quad \frac{m}{n}<x<\frac{m+1}{n}. \qquad (4)$$

It follows, from the nature of the process by which $\frac{m}{n}$ is obtained, that $b^{\frac{m}{n}}$ increases, but remains always *less than* a, and therefore becomes more and more nearly equal to a. For the same reason, $b^{\frac{m+1}{n}}$ decreases, but is always *greater than* a, and therefore becomes more and more nearly equal to a.

It will now be proved that as n increases beyond any assigned number, however great, $b^{\frac{m}{n}}$ and $b^{\frac{m+1}{n}}$ differ from a by less than any assigned number, however small.

For, $b^{\frac{m}{n}}$ and $b^{\frac{m+1}{n}}$ differ from each other, and hence from a, which lies between them, by less than any assigned number, however small.

At the same time, as was proved in Ch. XVIII., Art. 7, $\frac{m}{n}$ and $\frac{m+1}{n}$ differ from each other, and therefore from a common limit which lies between them, by less than any assigned number, however small. This common limit, therefore, is such a value of x as makes $b^x = a$.

We have, therefore, proved that for given values of a and b, when $b > 1$ and a is positive, there is a definite value of x which satisfies the equation

$$b^x = a.$$

(ii.) When $b < 1$ and positive. Then $\frac{1}{b} > 1$ and positive. Therefore, by (i.) there is a value of x such that

$$\left(\frac{1}{b}\right)^x = a.$$

But, $$\left(\frac{1}{b}\right)^x = b^{-x}.$$

Therefore, there is always a value of x such that $b^x = a$, when $b < 1$ and positive.

Logarithms.

9. The value of x which satisfies the equation $b^x = a$ is called the *logarithm of a to the base b.*

The Logarithm of a given number a to a given base b is, therefore, the exponent of the power to which the base b must be raised to produce the number a.

E.g., since $2^3 = 8$, 3 is the logarithm of 8 to the base 2; since $10^2 = 100$, 2 is the logarithm of 100 to the base 10.

10. The relation $b^x = a$ is also written $x = \log_b a$, read *x is the logarithm of a to the base b.* Thus,

$$2^3 = 8 \quad \text{and } 3 = \log_2 8,$$

$$10^2 = 100 \text{ and } 2 = \log_{10} 100,$$

are equivalent ways of expressing one and the same relation.

11. The theory of logarithms is based upon the idea of representing all positive numbers, in their natural order, as powers of one and the same base.

Thus, 4, 8, 16, 32, 64, etc., can all be expressed as powers of a common base 2; as $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, etc. Since, also, all the numbers intermediate between those given above can be expressed as powers of 2, the exponents of these powers are the logarithms of the corresponding numbers.

The logarithms of all positive numbers to a given base form what is called a **System of Logarithms**. The base is then called the *base of the system*.

12. Neither the number 1, nor any negative number, can be taken as the base of a system of logarithms. For, since any power of 1 is 1, it is evident that any other number than 1 cannot be represented as a power of 1.

It is also impossible to represent all positive numbers as real powers of a given negative number.

It follows from Art. 8 that any positive number except 1 may be taken as the base of a system of logarithms.

13. The following properties of logarithms evidently follow from the properties of powers.

(i.) *The logarithm of* 1 *to any base is* 0. For $b^0 = 1$, or $\log_b 1 = 0$.

(ii.) *The logarithm of the base itself is* 1. For $b^1 = b$, or $\log_b b = 1$.

The following properties of logarithms hold when the base is greater than 1.

(iii.) *The logarithm of an infinite is an infinite.* For $b^\infty = \infty$, or $\log_b \infty = \infty$.

(iv.) *The logarithm of* 0 *is a negative infinite.* For $b^{-\infty} = 0$, or $\log_b 0 = -\infty$.

(v.) *The logarithm is positive or negative, according as the number is* $>$ *or* < 1.

For any positive number > 1 lies between 1 and $+\infty$. Therefore its logarithm lies between 0 and $+\infty$, and is positive. Any positive number < 1 lies between 0 and 1; therefore its logarithm lies between $-\infty$ and 0, and is negative.

14. The following relation will be found useful:

$$b^{\log_b a} = a.$$

For let $\log_b a = x$, or $b^x = a$.

Substituting in the last relation $\log_b a$ for x, we have $b^{\log_b a} = a$.

The truth of this relation is self-evident. It asserts that *the logarithm of a, to the base b*, is the exponent $\log_b a$; but the italicized words are just the words for which the expression $\log_b a$ stands.

EXERCISES I.

Express the following relations in the language of logarithms:

1. $5^2 = 25$. **2.** $2^5 = 32$. **3.** $7^3 = 343$. **4.** $3^7 = 2187$.
5. $4^4 = 256$. **6.** $3^3 = 27$. **7.** $5^3 = 125$. **8.** $8^3 = 512$.

Express the following relations in terms of powers:

9. $\log_3 81 = 4$. **10.** $\log_9 81 = 2$. **11.** $\log_4 64 = 3$. **12.** $\log_2 64 = 6$.
13. $\log_8 512 = 3$. **14.** $\log_3 729 = 6$. **15.** $\log_{\frac{1}{2}} 16 = -4$. **16.** $\log_{10} .001 = -3$.

Determine the values of the following logarithms:

17. $\log_2 32$. **18.** $\log_{\frac{1}{2}} 128$. **19.** $\log_2 .5$. **20.** $\log_2 .25$.
21. $\log_4 64$. **22.** $\log_{64} 8$. **23.** $\log_2 .125$. **24.** $\log_5 .04$.
25. $\log_{729} 3$. **26.** $\log_{3125} 5$. **27.** $\log_{2401} 7$. **28.** $\log_2 .03125$.
29. $\log_4 .15625$. **30.** $\log_{27} \frac{1}{243}$. **31.** $\log_{64} .5$. **32.** $\log_{32} .125$.

To the base 16, what numbers have the following logarithms?

33. 0. **34.** $\frac{1}{2}$. **35.** -2. **36.** $\frac{3}{2}$. **37.** $-\frac{1}{4}$.

Solve the following equations:

38. $\log_2 x = 3$. **39.** $\log_2 x = .5$. **40.** $\log_2 x = \frac{1}{16}$.
41. $\log_x 9 = 2$. **42.** $\log_x 27 = -3$. **43.** $\log_x 8 = \frac{1}{2}$.

Principles of Logarithms.

15. *The logarithm of a product is equal to the sum of the logarithms of its factors;* or,

$$\log_b (m \times n) = \log_b m + \log_b n.$$

Let $\log_b m = x$ and $\log_b n = y$;

then $b^x = m$ and $b^y = n$, and therefore, $mn = b^x b^y = b^{x+y}$.

Translated into the language of logarithms, this result reads

$$\log_b(mn) = x + y.$$

But $x = \log_b m$ and $y = \log_b n$,

and consequently

$$\log_b(mn) = \log_b m + \log_b n,$$

for all positive values of b.

This result may be readily extended to a product of any number of factors. For,

$$\log_b(mnp) = \log_b(mn) + \log_b p = \log_b m + \log_b n + \log_b p.$$

And, in like manner, for any number of factors.

E.g. Given $\log_2 32 = 5$, and $\log_2 64 = 6$; what is the logarithm of 2048 to the base 2?

Since $2048 = 32 \cdot 64$, we have

$$\log_2 2048 = \log_2 32 + \log_2 64 = 5 + 6 = 11.$$

16. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor;* or,

$$\log_b (m \div n) = \log_b m - \log_b n.$$

Let $\log_b m = x$ and $\log_b n = y$;

then $b^x = m$ and $b^y = n$, and therefore $m \div n = b^x \div b^y = b^{x-y}$.

In the language of logarithms the last equation is

$$\log_b (m \div n) = x - y = \log_b m - \log_b n,$$

for all positive values of b.

E.g. Given $\log_3 3 = 1$ and $\log_3 2187 = 7$, what is the logarithm of 729 to the base 3?

Since $729 = \frac{2187}{3}$,

we have $\log_3 729 = \log_3 2187 - \log_3 3 = 7 - 1 = 6.$

17. Both m and n may be products, or the quotient of two numbers.

E.g., $$\log_{10} \frac{4 \times 5}{9 \times 8} = \log_{10} (4 \times 5) - \log_{10} (9 \times 8)$$
$$= \log_{10} 4 + \log_{10} 5 - \log_{10} 9 - \log_{10} 8.$$

18. The logarithm of the reciprocal of any number is the opposite of the logarithm of the number.

For, $\log_b \frac{1}{n} = \log_b 1 - \log_b n$

$= -\log_b n$, since $\log_b 1 = 0$.

E.g., $\log_2 4 = 2$, and $\log_2 \frac{1}{4} = -2$.

19. *The logarithm of any power, integral or fractional, of a number is equal to the logarithm of the number multiplied by the exponent of the power;* or,

$$\log (m^p) = p \log m.$$

Let $\log_b m = x$, then $b^x = m$.

Raising both sides of the last equation to the pth power, we have $b^{px} = m^p$, or $\log_b (m^p) = px = p \log_b m$.

E.g., if $\log_5 25 = 2$, what is $\log_5 (25)^3$?

We have $\log_5 (25)^3 = 3 \log_5 25 = 3 \times 2 = 6$.

20. When the exponent is a positive fraction whose numerator is 1, this principle may be conveniently stated thus:

The logarithm of a root of a number is the logarithm of the number divided by the index of the root.

For, $\log_b (m^{\frac{1}{q}}) = \frac{1}{q} \log m = \frac{\log_b m}{q}$.

E.g., If $\log_7 2401 = 4$, what is $\log_7 \sqrt{2401}$?

We have

$$\log_7 \sqrt{2401} = \frac{1}{2} \log_7 2401 = \frac{1}{2} \cdot 4 = 2.$$

21. It can readily be seen from the preceding principles and examples that if the logarithms of all numbers to any one base are given, certain numerical calculations can be greatly simplified by replacing the operations of multiplication and division by those of addition and subtraction, and the operations of involution and evolution by those of multiplication and division.

EXERCISES II.

Express the following logarithms in terms of $\log a$, $\log b$, $\log c$, and $\log d$:

1. $\log \frac{abc}{d}$. **2.** $\log \frac{d}{abc}$. **3.** $\log \frac{ac^2}{bd^2}$. **4.** $\log \left(\frac{ac}{bd}\right)^2$.

5. $\log a^{\frac{5}{6}} d^{-\frac{2}{3}} \sqrt{b} \sqrt{c}$. **6.** $\log \frac{2ab^2}{3c\sqrt{d}}$. **7.** $\log \frac{a^{-2}b^{\frac{3}{2}}}{\sqrt{(c^5 d^{-8})}}$.

Express the following sums of logarithms as logarithms of products and quotients.

8. $\log a + \log b - \log c$. **9.** $\log a - (\log b + \log c)$.

10. $3 \log a - \frac{1}{2} \log (b + c)$. **11.** $\frac{1}{2} \log (1 - x) + \frac{3}{2} \log (1 + x)$.

12. $2 \log \frac{a}{b} + 3 \log \frac{b}{a}$. **13.** $2 \log a - \frac{2}{3} \log b + \frac{1}{2} \log c$.

Given $\log_{10} 2 = .3010$, $\log_{10} 3 = .4771$, $\log_{10} 5 = .6990$, $\log_{10} 7 = .8451$, find the values of the following logarithms, to the base 10:

14. $\log 5$. **15.** $\log 6$. **16.** $\log 8$. **17.** $\log 9$.

18. $\log 12$. **19.** $\log 36$. **20.** $\log 108$. **21.** $\log 4\frac{1}{2}$.

22. $\log 2\frac{2}{3}$. **23.** $\log 5\frac{5}{6}$. **24.** $\log 5\frac{1}{7}$. **25.** $\log 360$.

26. $\log 3072$. **27.** $\log 3500$. **28.** $\log 5880$.

29. $\log \sqrt{72}$. **30.** $\log \sqrt{180}$. **31.** $\log \sqrt{1715}$.

32. $\log \frac{\sqrt[5]{490}}{\sqrt[3]{96}}$. **33.** $\log \frac{\sqrt[6]{9\frac{3}{5}} \times \sqrt{105}}{\sqrt[3]{72} \times \sqrt[4]{8\frac{2}{5}}}$. **34.** $\log \frac{(2\frac{2}{3})^{-2}}{(11\frac{2}{3})^{\frac{3}{2}}}$.

Find the values of the following fractions:

35. $\frac{\log 72}{\log 9}$. **36.** $\frac{\log 625}{\log 125}$. **37.** $\frac{\log 243}{\log 3}$.

38. Prove that the ratio of the logarithms of two numbers is the same for all bases.

Systems of Logarithms.

22. The two most important systems of logarithms are:

(i.) The system whose base is 10. This system was introduced, in 1615, by the Englishman, Henry Briggs.

Logarithms to the base 10 are called **Common, or Briggs's Logarithms.**

(ii.) The system whose base is the sum of the following infinite series,

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots.$$

The value of this sum, which to seven places of decimals is 2.7182818, is denoted by the letter e.

Logarithms to the base e are called **Natural Logarithms**; sometimes also **Napierian Logarithms**, in honor of the inventor of logarithms, the Scotch Baron Napier, a contemporary of Briggs. Napier himself did not, however, introduce this system of logarithms.

These two systems are the only ones which have been generally adopted; the common system is used in practical calculations, the natural system in theoretical investigations. The reason that in all practical calculations the common system of logarithms is superior to other systems is because its base 10 is also the base of our decimal system of numeration.

The logarithms of most numbers are irrational, and thus approximate values are used.

Properties of Common Logarithms.

23. In the following articles the subscript denoting the base 10 will be omitted.

We now have

$$(a)\begin{cases}10^0 = 1, \text{ or } \log 1 = 0,\\ 10^1 = 10, \text{ or } \log 10 = 1;\\ 10^2 = 100, \text{ or } \log 100 = 2;\\ 10^3 = 1000, \text{ or } \log 1000 = 3;\end{cases}$$

$$\cdots\cdots\cdots$$

$$(b)\begin{cases}10^{-1} = .1, \text{ or } \log .1 = -1;\\ 10^{-2} = .01, \text{ or } \log .01 = -2;\\ 10^{-3} = .001, \text{ or } \log .001 = -3;\\ 10^{-4} = .0001, \text{ or } \log .0001 = -4;\end{cases}$$

$$\cdots\cdots\cdots$$

Evidently the logarithms of all positive numbers, except positive and negative integral powers of 10, consist of an integral and a decimal part. Thus, since $10^1 < 85 < 10^2$, we have $1 < \log 85 < 2$, or $\log 85 = 1 +$ *a decimal.*

24. The integral part of a logarithm is called its **Characteristic.**

The decimal part of a logarithm is called its **Mantissa.**

25. Since a number having one digit in its integral part, as 7.3, lies between 10^0 and 10^1, it follows from table (a) that its logarithm lies between 0 and 1, *i.e.*, is $0+$ *a decimal.* Since any number having two digits in its integral part, as 76.4, lies between 10^1 and 10^2, its logarithm lies between 1 and 2, that is, is $1 +$ *a decimal.* In general, since any number having n digits in its integral part lies between 10^{n-1} and 10^n, its logarithm lies between $n-1$ and n, *i.e.*, is $n - 1 +$ *a decimal.* We therefore have:

(i.) *The characteristic of the logarithm of a number greater than unity is positive, and is one less than the number of digits in its integral part.*

E.g., $\log 2756.3 = 3 +$ *a decimal.*

Since a number less than 1 having no cipher immediately following the decimal point lies between 10^0 and 10^{-1}, it follows from table (b), that its logarithm lies between 0 and -1, *i.e.*, is $-1 +$ *a positive decimal.* Since a number less than 1 having one cipher immediately following the decimal point lies between 10^{-1} and 10^{-2}, its logarithm lies between -1 and -2, *i.e.*, is $-2 +$ *a positive decimal.* In general, since a number less than 1 having n ciphers immediately following the decimal point lies between 10^{-n} and $10^{-(n+1)}$, its logarithm lies between $-n$ and $-(n+1)$, *i.e.*, is $-(n+1) +$ *a positive decimal.* We therefore have:

(ii.) *The characteristic of the logarithm of a number less than* 1 *is negative, and is numerically one greater than the number of ciphers immediately following the decimal point.*

E.g., $\log .00035 = -4 +$ *a decimal fraction.*

It follows conversely from (i.) and (ii.):

(iii.) *If the characteristic of a logarithm be $+n$, there are $n+1$ digits in the integral part of the corresponding number.*

(iv.) *If the characteristic of a logarithm be $-n$, there are $n-1$ ciphers immediately following the decimal point of the corresponding number.*

26. It has been found that $538 = 10^{2.7308}$ to four decimal places, or $\log 538 = 2.7308$. We also have

$$\log .0538 = \log \tfrac{538}{10000} = \log 538 - \log 10000 = 2.7308 - 4$$
$$= .7308 - 2;$$

$$\log 5.38 = \log \tfrac{538}{100} = \log 538 - \log 100 = 2.7308 - 2$$
$$= .7308;$$

$$\log 53800 = \log (538 \times 100) = \log 538 + \log 100$$
$$= 2.7308 + 2 = 4.7308.$$

These examples illustrate the following principle:

If two numbers differ only in the position of their decimal points, their logarithms have different characteristics but the same positive mantissa.

If n denote the number of places through which the decimal point has been moved in a given number a, we have

$$\log (a\, 10^n) = \log a + n \log 10 = \log a + n,$$

and $$\log (a \div 10^n) = \log a - n \log 10 = \log a - n,$$

since moving the decimal point a given number of places to the right or left is equivalent to multiplying or dividing by a power of 10.

27. The characteristic and the mantissa of a number less than 1 may be connected by the decimal point, if the sign (—) be written over the characteristic to indicate that the characteristic only is negative, and not the entire number.

Thus instead of $\log .00709 = .8506 - 3 = -3 + .8506$, we may write $\bar{3}.8506$; this must be distinguished from the expression -3.8506, in which the integer and the decimal are both negative. Similarly,

$$\log .082 = \bar{2}.9138, \text{ while } \log 820 = 2.9138.$$

28. The logarithms, to the base 10, of a set of consecutive positive numbers have been computed. The student is referred to Tables of Logarithms, and to works on Trigonometry, for specific directions in the use of logarithms.

29. The following relation is sometimes useful:

$$\log_b a \cdot \log_a b = 1.$$

If $\log_b a = x$ and $\log_a b = y$,

we have $b^x = a$ (1) and $a^y = b$. (2)

Raising (1) to the yth power, we obtain

$$b^{xy} = a^y, = b.$$

Therefore $xy = 1$, or $\log_b a \cdot \log_a b = 1$.

30. If the logarithms of any system (*i.e.*, to any base) have been calculated, the logarithms of any other system (*i.e.*, to any other base) can be easily obtained from them.

Let $\log_a N = x$ and $\log_b N = y$;

then $N = a^x$ and $N = b^y$.

Consequently, $a^x = b^y$.

The last equation is equivalent to

$$x = \log_a b^y = y \cdot \log_a b;$$

therefore, $y = \dfrac{x}{\log_a b} = \dfrac{1}{\log_a b} \cdot x$.

Hence, to transform the logarithm of a number from base a to base b, divide it by $\log_a b$, *i.e.*, the logarithm of the new base to the old base.

It follows that if the logarithms of any system (say, to base a) have been calculated, the logarithms of any other system (say to base b) are obtained by multiplying each logarithm of the first system by the constant number $\dfrac{1}{\log_a b}$. The latter number is called the **Modulus** of the system b with respect to the system a.

31. If the natural logarithm of a number N be denoted by n and its common logarithm by m, then

$$N = e^n = 10^m.$$

From this equation we obtain

$$n \log_{10} e = m \log_{10} 10 = m,$$

or $n = \dfrac{1}{\log_{10} e} \cdot m = \dfrac{1}{\log_{10} 2.7182818 \cdots} \cdot m$.

The value of $\log_{10} 2.7182818 \cdots$ is known to be .4342945, to seven places of decimals.

Consequently, $$n = \frac{1}{.4342945} \times m = 2.3025851 \times m,$$

or $$\log_e N = 2.3025851 \log_{10} N.$$

The number 2.3025851 is therefore the modulus of the natural system with respect to the common system of logarithms, *i.e.*, is the number by which each logarithm of the common system must be multiplied in order to give the logarithms of the natural system.

Similarly, $$\log_{10} N = 0.4342945 \log_e N,$$

or, 4342945 is the modulus of the common system with respect to the natural system of logarithms.

EXERCISES III.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find the logarithms of the following numbers:

1. .2. **2.** .002. **3.** 500. **4.** 7000. **5.** .0003.

6. $\frac{20}{.3}$. **7.** $\frac{.3}{20}$. **8.** $\frac{.007}{.002}$. **9.** $\frac{20^7}{.02^{19}}$. **10.** $\frac{.005^8}{.007^2}$.

Determine the number of integral places in the following powers:

11. 2^{100}. **12.** 7^{60}. **13.** 5^{1000}. **14.** 294^{999}.

Exponential and Logarithmic Equations.

32. **An Exponential Equation** is an equation in which the unknown number appears as an exponent of a known or an unknown number; as, $a^x = b$.

A Logarithmic Equation is an equation in which the logarithm of the unknown number, or of an expression containing the unknown number, enters; as, $\log(x + 1) = 2$.

The solutions of certain forms of exponential and logarithmic equations will be considered.

Exponential Equations.

33. Ex. **1.** Solve the equation $3^x = 9$.

Taking logarithms, $x \log 3 = \log 9 = 2 \log 3.$

Hence $x = 2.$

This result could have been obtained by inspection, by writing $3^x = 3^2$.

Ex. **2.** Solve the equation $3^x = 5$.

Taking the logarithms of both sides of the equation, we have

$$x \log 3 = \log 5, \quad \text{or} \quad x = \frac{\log 5}{\log 3} = \frac{0.6990}{0.4771} = 1.465.$$

Ex. 3. Solve the equation $9^{\sqrt{(x+1)}} = 27 \cdot 3^{\sqrt{(x+1)}}$.

Replacing 9 by 3^2, and 27 by 3^3, we obtain

$$3^{2\sqrt{(x+1)}} = 3^3 \cdot 3^{\sqrt{(x+1)}} = 3^{3+\sqrt{(x+1)}}.$$

Hence $$2\sqrt{(x+1)} = 3 + \sqrt{(x+1)},$$

or $$x = 8.$$

Ex. 4. Solve the equation $4 \cdot 5^{x+1} - 5^x = 95$.

Since $$5^{x+1} = 5 \cdot 5^x,$$

we have $$4 \cdot 5 \cdot 5^x - 5^x = 95,$$

or $$19 \cdot 5^x = 95;$$

whence $$5^x = 5, \text{ or } x = 1.$$

Ex. 5. Solve the equation $10 \cdot 2^x - 2^{2x} = 16$.

Since $2^{2x} = (2^x)^2$, we solve the equation as an equation in 2^x as the unknown number. Replacing 2^x by y, we obtain

$$y^2 - 10y = -16;$$

whence $$y = 8 \text{ and } 2.$$

We therefore have the two exponential equations:

$$2^x = 8, \text{ or } x = 3,$$

and $$2^x = 2, \text{ or } x = 1.$$

Ex. 6. Solve the equations $2^x \cdot 3^y = 18, \quad (1)$

$$5^x \cdot 7^y = 245. \quad (2)$$

Taking logarithms, we obtain from (1)

$$x \log 2 + y \log 3 = \log 18, \quad (3)$$

and from (2) $$x \log 5 + y \log 7 = \log 245. \quad (4)$$

From (3) and (4), we have

$$x = \frac{\log 7 \cdot \log 18 - \log 3 \cdot \log 245}{\log 7 \cdot \log 2 - \log 3 \cdot \log 5} = 1,$$

$$y = \frac{-\log 5 \cdot \log 18 + \log 2 \cdot \log 245}{\log 7 \cdot \log 2 - \log 3 \cdot \log 5} = 2.$$

This example could also have been solved by inspection.

Since $$18 = 2 \cdot 3^2 \text{ and } 245 = 5 \cdot 7^2,$$

we have $$2^x \cdot 3^y = 2 \cdot 3^2, \quad (5)$$

$$5^x \cdot 7^y = 5 \cdot 7^2. \quad (6)$$

Assuming tentatively $x = 1$ and $y = 2$ in equation (5), we see that these values also satisfy equation (6).

Logarithmic Equations.

34. Ex. **1.** Solve the equation $\frac{1}{2}\log(x-9)+\log\sqrt{(2x-1)}=1.$

By the principles of logarithms, we obtain successively

$$\log\sqrt{(x-9)}+\log\sqrt{(2x-1)}=\log 10,$$
$$\log\sqrt{[(x-9)(2x-1)]}=\log 10.$$

Therefore $$\sqrt{[(x-9)(2x-1)]}=10,$$
or $$2x^2-19x+9=100.$$

The roots of this equation are 13 and $-\frac{7}{2}$.

Ex. **2.** Solve the equation

$$\log(x+12)-\log x=0.8451+\log(6-6x).$$

By the principles of logarithms,

$$\log\frac{x+12}{x}=\log 7(6-5x),\text{ since } 0.8451=\log 7.$$

Consequently $$\frac{x+12}{x}=42-35x,$$
or $$x+12=42x-35x^2.$$

The roots of this equation are $\frac{4}{7}$ and $\frac{21}{35}$.

Ex. **3.** Solve the equation $x^{\log x}=100x.$

Taking logarithms, we obtain

$$(\log x)^2=\log 100+\log x,$$
or $$(\log x)^2-\log x=2.$$

Solving this equation as a quadratic in $\log x$, we obtain

$$\log x=\ \ 2,\text{ or } x=100;$$
$$\log x=-1,\text{ or } x=\tfrac{1}{10}.$$

EXERCISES IV.

Solve the following exponential equations:

1. $2^x=64.$

2. $3^x=81.$

3. $2^{x-1}=.5^{2x-5}.$

4. $(1\frac{1}{3})^7=.75^{x-3}.$

5. $4^{3x-1}=.5^{x-5}.$

6. $4^x=8.$

7. $8^x=32.$

8. $5^x=(\sqrt{5})^{-1}.$

9. $4^{x+1}=8\cdot 2^{x+2}.$

10. $25^{3x-1}=625\cdot 5^{x+3}.$

11. $7^{\sqrt{(x-3)}}=343-49^{\sqrt{(x-3)}}.$

12. $27^{\sqrt{(x-3)}}=(\sqrt{3})^{2\sqrt{(x+3)}}.$

13. $\sqrt{a^{11-x}}=a^{8-x}.$

14. $\sqrt[3]{a^{x+2}}=\sqrt{a^{x-3}}.$

15. $\sqrt[3]{a^{x+2}}=\sqrt{a^{x-3}}.$

16. $4^x-6\cdot 2^x+8=0.$

17. $9^x+243=36\cdot 3^x.$

Given log 2 = .3010, log 3 = .4771, log 5 = .6990, log 7 = .8451, solve the following equations:

18. $10^x = 5.$ **19.** $5^x = 10.$ **20.** $100^x = 8.$

21. $1000^x = 6.3.$ **22.** $(\frac{3}{4})^x = \frac{2}{3}.$ **23.** $(\frac{4}{5})^x = \frac{1}{5}.$

24. $\begin{cases} 3^x \cdot 5^y = 75, \\ 2^x \cdot 7^y = 98. \end{cases}$ **25.** $\begin{cases} 14^x \cdot 8^y = 896, \\ 5^x \cdot 9^y = 405. \end{cases}$ **26.** $\begin{cases} x^{2y} = y^{2x}, \\ y^2 = x^3. \end{cases}$

Solve the following logarithmic equations:

27. $\log x + \log(x + 3) = 1.$ **28.** $\log 4 + 2 \log x = 2.$

29. $\log 8 + 3 \log x = 3.$ **30.** $2 \log x = 1 + \log(x + \frac{11}{10}).$

31. $\log \sqrt{(7x + 5)} + \log \sqrt{(2x + 3)} = 1 + \log \frac{9}{2}.$

32. $\log(7 - 9x)^2 + \log(3x - 4)^2 = 2.$

33. $\log(x + \sqrt{x}) + \log(x - \sqrt{x}) = \log 4 + \log x^3 - \log x.$

34. $\dfrac{\log x^2}{\log(3x - 16)} = 2.$ **35.** $\dfrac{\log(2x-3)}{\log(4x^2-15)} = \frac{1}{2}.$ **36.** $\dfrac{\log(35-x^3)}{\log(5-x)} = 3.$

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